# X. *The Effect produced by an Obstacle on a Train of Electric Waves.*

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INTEGRALS of the equations of propagation of electric disturbances in terms of the electric and magnetic forces tangential to any surface enclosing the sources of the disturbances have been already obtained.* It is proposed in what follows to apply these expressions to obtain the effect of an obstacle on a train of electric waves. The effect of the obstacle can be represented by a distribution of sources throughout the space occupied by the obstacle, and the determination of this distribution or, as appears from the investigation referred to above, the determination of the electric and magnetic forces tangential to the surface of the obstacle due to this distribution of sources, constitutes the solution of the problem. If $X'$, $Y'$, $Z'$, $\alpha'$, $\beta'$, $\gamma'$ denote the components of the electric and magnetic forces respectively at the point $\xi$, $\eta$, $\zeta$ on the surface of the obstacle due to the distribution of sources inside it which represents its effect, $X'_1$, $Y'_1$, $Z'_1$, $\alpha'_1$, $\beta'_1$, $\gamma'_1$ denote the values of these quantities when $t-r/V$ is substituted for $t$, where $V$ is the velocity of propagation of the disturbances, and $r = \{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2\}^{1/2}$ is the distance of any point $x$, $y$, $z$ from the point $\xi$, $\eta$, $\zeta$, and $l$, $m$, $n$ are the direction cosines of the normal to the surface at the point $\xi$, $\eta$, $\zeta$ drawn into the space external to the obstacle, the components of the magnetic force at any point outside the obstacle due to the obstacle are given by

$$\dot{\alpha} = \frac{1}{4\pi}\frac{\partial}{\partial t}\iint\left[\frac{\partial}{\partial y}\frac{\bar{\gamma}}{r} - \frac{\partial}{\partial z}\frac{\bar{\beta}}{r}\right]dS - \frac{1}{4\pi}\iint\left[\frac{\partial^2}{\partial x^2}\frac{\bar{X}}{r} + \frac{\partial^2}{\partial x\partial y}\frac{\bar{Y}}{r} + \frac{\partial^2}{\partial x\partial z}\frac{\bar{Z}}{r} - \frac{1}{V^2}\frac{\partial^2}{\partial t^2}\frac{\bar{X}}{r}\right]dS,$$

$$\dot{\beta} = \frac{1}{4\pi}\frac{\partial}{\partial t}\iint\left[\frac{\partial}{\partial z}\frac{\bar{\alpha}}{r} - \frac{\partial}{\partial x}\frac{\bar{\gamma}}{r}\right]dS - \frac{1}{4\pi}\iint\left[\frac{\partial^2}{\partial x\partial y}\frac{\bar{X}}{r} + \frac{\partial^2}{\partial y^2}\frac{\bar{Y}}{r} + \frac{\partial^2}{\partial y\partial z}\frac{\bar{Z}}{r} - \frac{1}{V^2}\frac{\partial^2}{\partial t^2}\frac{\bar{Y}}{r}\right]dS,$$

$$\dot{\gamma} = \frac{1}{4\pi}\frac{\partial}{\partial t}\iint\left[\frac{\partial}{\partial x}\frac{\bar{\beta}}{r} - \frac{\partial}{\partial y}\frac{\bar{\alpha}}{r}\right]dS - \frac{1}{4\pi}\iint\left[\frac{\partial^2}{\partial x\partial z}\frac{\bar{X}}{r} + \frac{\partial^2}{\partial y\partial z}\frac{\bar{Y}}{r} + \frac{\partial^2}{\partial z^2}\frac{\bar{Z}}{r} - \frac{1}{V^2}\frac{\partial^2}{\partial t^2}\frac{\bar{Z}}{r}\right]dS,$$

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* MACDONALD, "Electric Waves," 1902, pp. 16–17, 'Proc. Lond. Math. Soc.,' 1911.

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and the components of the electric force at the point $x$, $y$, $z$ are given by

$$\dot{\mathrm{X}} = \frac{\mathrm{V}^2}{4\pi}\iint\left[\frac{\partial^2}{\partial x^2}\frac{\bar{\alpha}}{r}+\frac{\partial^2}{\partial x\partial y}\frac{\bar{\beta}}{r}+\frac{\partial^2}{\partial x\partial z}\frac{\bar{\gamma}}{r}-\frac{1}{\mathrm{V}^2}\frac{\partial^2}{\partial t^2}\frac{\bar{\alpha}}{r}\right]d\mathrm{S}+\frac{1}{4\pi}\frac{\partial}{\partial t}\iint\left[\frac{\partial}{\partial y}\frac{\bar{\mathrm{Z}}}{r}-\frac{\partial}{\partial z}\frac{\bar{\mathrm{Y}}}{r}\right]d\mathrm{S},$$

$$\dot{\mathrm{Y}} = \frac{\mathrm{V}^2}{4\pi}\iint\left[\frac{\partial^2}{\partial x\partial y}\frac{\bar{\alpha}}{r}+\frac{\partial^2}{\partial y^2}\frac{\bar{\beta}}{r}+\frac{\partial^2}{\partial y\partial z}\frac{\bar{\gamma}}{r}-\frac{1}{\mathrm{V}^2}\frac{\partial^2}{\partial t^2}\frac{\bar{\beta}}{r}\right]d\mathrm{S}+\frac{1}{4\pi}\frac{\partial}{\partial t}\iint\left[\frac{\partial}{\partial z}\frac{\bar{\mathrm{X}}}{r}-\frac{\partial}{\partial x}\frac{\bar{\mathrm{Z}}}{r}\right]d\mathrm{S},$$

$$\dot{\mathrm{Z}} = \frac{\mathrm{V}^2}{4\pi}\iint\left[\frac{\partial^2}{\partial x\partial z}\frac{\bar{\alpha}}{r}+\frac{\partial^2}{\partial y\partial z}\frac{\bar{\beta}}{r}+\frac{\partial^2}{\partial z^2}\frac{\bar{\gamma}}{r}-\frac{1}{\mathrm{V}^2}\frac{\partial^2}{\partial t^2}\frac{\bar{\gamma}}{r}\right]d\mathrm{S}+\frac{1}{4\pi}\frac{\partial}{\partial t}\iint\left[\frac{\partial}{\partial x}\frac{\bar{\mathrm{Y}}}{r}-\frac{\partial}{\partial y}\frac{\bar{\mathrm{X}}}{r}\right]d\mathrm{S},$$

where

$$\bar{\alpha} = m\gamma'_1-n\beta'_1, \qquad \bar{\beta} = n\alpha'_1-l\gamma'_1, \qquad \bar{\gamma} = l\beta'_1-m\alpha'_1,$$

$$\bar{\mathrm{X}} = m\mathrm{Z}'_1-n\mathrm{Y}'_1, \qquad \bar{\mathrm{Y}} = n\mathrm{X}'_1-l\mathrm{Z}'_1, \qquad \bar{\mathrm{Z}} = l\mathrm{Y}'_1-m\mathrm{X}'_1,$$

and the integrations are taken over the surface of the obstacle. The quantities $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ are the components of the electric current distribution on the surface which would produce the tangential magnetic force on the surface due to the distribution of sources inside it, and the quantities $\bar{\mathrm{X}}$, $\bar{\mathrm{Y}}$, $\bar{\mathrm{Z}}$ are the components of the magnetic current distribution on the surface which would produce the tangential electric force on the surface due to the distribution of sources inside it. The waves incident on the surface can be represented as the effect of a distribution of Hertzian oscillators, and it will therefore be sufficient to consider the effect of a Hertzian oscillator situated at a point, O, outside the surface and emitting waves of a definite wave-length, $2\pi/\kappa$. Now, the conditions to be satisfied at the surface of the obstacle are linear relations involving the components of the electric and magnetic forces, and therefore the integrands in the above expressions for X, Y, Z, $\alpha$, $\beta$, $\gamma$ will each contain the factor $e^{-\iota\kappa(r+\Sigma\epsilon r')}$ where $r$ is the distance of the point $(x, y, z)$ from the point $(\xi, \eta, \zeta)$ on the surface, and the quantities $r'$ are other distances. The remaining factors of the integrands will be non-oscillatory unless the surface has corrugations on it, for which the interval between successive corrugations is comparable with the wave-length $2\pi/\kappa$ of the oscillations. The principal parts of the expressions for X, Y, Z, $\alpha$, $\beta$, $\gamma$ are therefore contributed by the portion or portions of the surface in the neighbourhood of the point or points for which $r+\Sigma\epsilon r'$ is stationary. If the wave-length of the oscillations is small compared with both principal radii of curvature of the surface at a point at which $r+\Sigma\epsilon r'$ is stationary, the corresponding principal parts of X, Y, Z, $\alpha$, $\beta$, $\gamma$ are the same for a point, P, very near to this point Q on the surface as if the point Q were situated on a plane surface. Hence the principal parts of X, Y, Z, $\alpha$, $\beta$, $\gamma$ at points on the surface are related to the principal parts of the components of the electric and magnetic forces in the incident waves in the same way as if the surface were plane, but it should be observed that the incident waves to be taken into account may include waves due to the obstacle in addition to the waves due to the external oscillator. The electric and magnetic current distributions to be assumed on the surface of the obstacle are then the same at each point of the surface as if that point were on an infinite plane surface coinciding with the tangent plane to the surface of the obstacle at the point, and the principal parts of X, Y, Z, $\alpha$, $\beta$, $\gamma$ at any point $x$, $y$, $z$ determined on this assumption are the leading terms in the asymptotic expressions for these quantities.

### *A Perfectly Conducting Obstacle.*

When the obstacle is a perfect conductor the condition to be satisfied at the surface is that the electric force tangential to the surface vanishes; it follows from the above that the principal part of the effect of the obstacle is obtained by assuming an electric current distribution on the parts of the surface on which waves are incident which is double the electric current distribution that would produce the magnetic force tangential to the surface in the incident waves, a zero electric current distribution on the parts of the surface on which no waves are incident, and a zero magnetic current distribution on all parts of the surface.

Taking first the case where the perfectly conducting obstacle is a convex solid, the waves incident on any part of the surface are due to the oscillator outside it, and, if M is the electric current distribution at any point on the surface which would produce the magnetic force tangential to the surface in the incident waves, the electric current distribution to be assumed at this point is 2M, if the point is on the part of the surface of the obstacle next the oscillator, and zero if it is on the part of the surface remote from the oscillator.

Let the origin of the co-ordinates be at the point O, the axis of the oscillator being the axis of $z$. The components of the magnetic force $(\alpha', \beta', \gamma')$ at the point $(\xi, \eta, \zeta)$ due to the oscillator are given by

$$\alpha' = \frac{\kappa}{\mathrm{V}}\frac{\partial}{\partial\eta}\frac{e^{\iota\kappa(\mathrm{V}t-r_1)}}{r_1}, \qquad \beta' = -\frac{\kappa}{\mathrm{V}}\frac{\partial}{\partial\xi}\frac{e^{\iota\kappa(\mathrm{V}t-r_1)}}{r_1}, \qquad \gamma' = 0,$$

where

$$r_1^2 = \xi^2+\eta^2+\zeta^2.$$

If $l$, $m$, $n$ are the direction cosines of the outward drawn normal to the surface at the point $(\xi, \eta, \zeta)$ on it, the components of the electric current distribution M which would produce the magnetic force tangential to the surface are

$$m\gamma'-n\beta', \qquad n\alpha'-l\gamma', \qquad l\beta'-m\alpha',$$

and the above hypothesis is equivalent to assuming an electric current distribution on the parts of the surface on which waves are incident whose components are

$$2(m\gamma'-n\beta'), \qquad 2(n\alpha'-l\gamma'), \qquad 2(l\beta'-m\alpha'),$$

a zero electric current distribution on the parts of the surface in the geometrical shadow, and a zero magnetic current distribution on all parts of the surface. The components of the magnetic force at the point $(x, y, z)$ due to this distribution are by the above

$$\alpha = \frac{1}{4\pi}\iint\left[\frac{\partial}{\partial y}\frac{\bar{\gamma}}{r} - \frac{\partial}{\partial z}\frac{\bar{\beta}}{r}\right] d\mathrm{S},$$

$$\beta = \frac{1}{4\pi}\iint\left[\frac{\partial}{\partial z}\frac{\bar{\alpha}}{r} - \frac{\partial}{\partial x}\frac{\bar{\gamma}}{r}\right] d\mathrm{S},$$

$$\gamma = \frac{1}{4\pi}\iint\left[\frac{\partial}{\partial x}\frac{\bar{\beta}}{r} - \frac{\partial}{\partial y}\frac{\bar{\alpha}}{r}\right] d\mathrm{S},$$

where

$$\bar{\alpha} = 2\,(m\gamma'_1 - n\beta'_1), \qquad \bar{\beta} = 2\,(n\alpha'_1 - l\gamma'_1), \qquad \bar{\gamma} = 2\,(l\beta'_1 - m\alpha'_1),$$

$$r^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2,$$

and the integration is taken over the portion of the surface on which waves from the oscillator are incident. Now

$$\alpha'_1 = \frac{\kappa\eta}{\mathrm{V}r_1}\frac{\partial}{\partial r_1}\frac{e^{\iota\kappa(\mathrm{V}t-r_1-r)}}{r_1}, \qquad \beta'_1 = -\frac{\kappa\xi}{\mathrm{V}r_1}\frac{\partial}{\partial r_1}\frac{e^{\iota\kappa(\mathrm{V}t-r_1-r)}}{r_1}, \qquad \gamma'_1 = 0,$$

whence

$$\bar{\alpha} = \frac{2\kappa}{\mathrm{V}}\frac{n\xi}{r_1}\frac{\partial}{\partial r_1}\frac{e^{\iota\kappa(\mathrm{V}t-r_1-r)}}{r_1}, \qquad \bar{\beta} = \frac{2\kappa}{\mathrm{V}}\frac{n\eta}{r_1}\frac{\partial}{\partial r_1}\frac{e^{\iota\kappa(\mathrm{V}t-r_1-r)}}{r_1}, \qquad \bar{\gamma} = -\frac{2\kappa}{\mathrm{V}}\frac{l\xi+m\eta}{r_1}\frac{\partial}{\partial r_1}\frac{e^{\iota\kappa(\mathrm{V}t-r_1-r)}}{r_1},$$

and therefore

$$\alpha = -\frac{\kappa}{2\pi\mathrm{V}}\iint\left[\frac{y-\eta}{r}\frac{l\xi+m\eta}{r_1} + \frac{z-\zeta}{r}\frac{n\eta}{r_1}\right]\frac{\partial^2}{\partial r\partial r_1}\frac{e^{\iota\kappa(\mathrm{V}t-r_1-r)}}{rr_1}\,d\mathrm{S},$$

$$\beta = \frac{\kappa}{2\pi\mathrm{V}}\iint\left[\frac{z-\zeta}{r}\frac{n\xi}{r_1} + \frac{x-\xi}{r}\frac{l\xi+m\eta}{r_1}\right]\frac{\partial^2}{\partial r\partial r_1}\frac{e^{\iota\kappa(\mathrm{V}t-r_1-r)}}{rr_1}\,d\mathrm{S},$$

$$\gamma = \frac{\kappa}{2\pi\mathrm{V}}\iint\frac{n\,(x\eta-y\xi)}{rr_1}\frac{\partial^2}{\partial r\partial r_1}\frac{e^{\iota\kappa(\mathrm{V}t-r_1-r)}}{rr_1}\,d\mathrm{S}.$$

The principal parts of these integrals are contributed by the elements near to the point for which the exponent of $e^{-\iota\kappa(r_1+r)}$ is stationary, that is, by the portion of the surface in the immediate neighbourhood of the point for which $r_1+r$ is stationary. The conducting surface being convex towards the point O, at which the oscillator is, if the point P $(x, y, z)$ is external to the tangent cone drawn from the point O to the conducting surface or internal to the tangent cone and on the same side of the conducting surface as the point O, the point for which $r_1+r$ is stationary is the point of contact Q of a prolate spheroid which has the points O and P as foci and touches the conducting surface, and, if the point P is internal to the tangent cone from O to the conducting surface and on the side of the conducting surface remote from the point O, the point for which $r_1+r$ is stationary is the point at which the straight line OP cuts the conducting surface nearest to the point O. The values of the principal parts of $\alpha$, $\beta$, $\gamma$ are therefore $\alpha_0$, $\beta_0$, $\gamma_0$, where

$$\alpha_0 = \frac{\kappa^3 \mathrm{I}}{2\pi \mathrm{V}} \{(l\xi_0+m\eta_0+n\zeta_0)(y-\eta_0)+n(\eta_0 z-\zeta_0 y)\} e^{\iota\kappa \mathrm{V}t}/\mathrm{R}^2\mathrm{R}_1^2,$$

$$\beta_0 = -\frac{\kappa^3 \mathrm{I}}{2\pi \mathrm{V}} \{(l\xi_0+m\eta_0+n\zeta_0)(x-\xi_0)+n(\xi_0 x-\zeta_0 x)\} e^{\iota\kappa \mathrm{V}t}/\mathrm{R}^2\mathrm{R}_1^2,$$

$$\gamma_0 = -\frac{\kappa^3 \mathrm{I}}{2\pi \mathrm{V}} n(\eta_0 x-\xi_0 y) e^{\iota\kappa \mathrm{V}t}/\mathrm{R}^2\mathrm{R}_1^2,$$

where $(\xi_0, \eta_0, \zeta_0)$ are the co-ordinates of the point Q for which $r_1+r$ is stationary, $l$, $m$, $n$ now denote the direction cosines of the outward drawn normal to the surface at the point Q, I is the principal value of the integral

$$\iint e^{-\iota\kappa(r_1+r)}\, d\mathrm{S},$$

and

$$\mathrm{R}^2 = (x-\xi_0)^2+(y-\eta_0)^2+(z-\zeta_0)^2, \qquad \mathrm{R}_1^2 = \xi_0^2+\eta_0^2+\zeta_0^2.$$

To calculate the value of I, it is convenient to choose for axes of reference the normal to the surface at the point Q $(\xi_0, \eta_0, \zeta_0)$ as the axis of $\zeta$, the tangent to the surface in the plane of incidence as the axis of $\xi$, and the perpendicular tangent as the axis of $\eta$. Let the equation of the surface referred to these axes be

$$2\zeta = \mathrm{A}\xi^2+2\mathrm{H}\xi\eta+\mathrm{B}\eta^2+\ldots,$$

let $(x_1, 0, z_1)$ be the co-ordinates of the point O, and let $(x_2, y_2, z_2)$ be the co-ordinates of the point P referred to these axes, then

$$r_1^2 = (\xi-x_1)^2+\eta^2+(\zeta-z_1)^2, \qquad r^2 = (\xi-x_2)^2+(\eta-y_2)^2+(\zeta-z_2)^2,$$

and further

$$r_1\frac{\partial r_1}{\partial \xi} = (\xi-x_1)+(\zeta-z_1)(\mathrm{A}\xi+\mathrm{H}\eta+\ldots),$$

$$r\,\frac{\partial r}{\partial \xi} = (\xi-x_2)+(\zeta-z_2)(\mathrm{A}\xi+\mathrm{H}\eta+\ldots),$$

$$r_1\frac{\partial r_1}{\partial \eta} = \eta+(\zeta-z_1)(\mathrm{H}\xi+\mathrm{B}\eta+\ldots),$$

$$r\,\frac{\partial r}{\partial \eta} = \eta-y_2+(\zeta-z_2)(\mathrm{H}\xi+\mathrm{B}\eta+\ldots).$$

Now, $r_1+r$ is stationary at the point Q for which

$$\xi = \eta = \zeta = 0,$$

therefore

$$\frac{x_1}{R_1}+\frac{x_2}{R} = 0, \qquad y_2 = 0;$$

whence

$$\frac{z_1^2}{R_1^2} = \frac{z_2^2}{R^2}, \qquad \text{that is} \qquad \frac{z_1}{R_1} = \pm\frac{z_2}{R}.$$

In this result the upper sign corresponds to the case where the point P is external to the tangent cone from the point O to the surface, or inside the tangent cone and on the same side of the surface as the point O; the lower sign corresponds to the case where the point **P** is inside the tangent cone and on the side of the surface remote from the point O, and in this case the point Q lies on the straight line OP. Considering first the case where OQP is a straight line, writing

$$x_1 = -R_1 \sin\phi, \qquad z_1 = -R_1\cos\phi,$$

it follows from the relations

$$z_1/R_1+z_2/R = 0, \qquad x_1/R_1+x_2/R = 0,$$

that

$$x_2 = R\sin\phi, \qquad y_2 = 0, \qquad z_2 = R\cos\phi;$$

hence

$$r_1^2 = R_1^2+2\xi R_1\sin\phi+2\zeta R_1\cos\phi+\xi^2+\eta^2+\zeta^2,$$

$$r^2 = R^2-2\xi R\sin\phi-2\zeta R_1\cos\phi+\xi^2+\eta^2+\zeta^2,$$

and therefore

$$r_1 = R_1+\xi\sin\phi+\zeta\cos\phi+\tfrac{1}{2}(\xi^2+\eta^2)/R_1-\tfrac{1}{2}\xi^2\sin^2\phi/R_1+\ldots,$$

$$r = R-\xi\sin\phi-\zeta\cos\phi+\tfrac{1}{2}(\xi^2+\eta^2)/R-\tfrac{1}{2}\xi^2\sin^2\phi/R+\ldots,$$

whence

$$r_1+r = R_1+R+\tfrac{1}{2}(\xi^2\cos^2\phi+\eta^2)(R_1^{-1}+R^{-1})+\ldots,$$

the remaining terms involving higher powers of $\xi$ and $\eta$.

Now

$$\iint e^{-\iota\kappa(r_1+r)}\,dS = \iint e^{-\iota\kappa(r_1+r)}\sqrt{\left\{1+\left(\frac{\partial\zeta}{\partial\xi}\right)^2+\left(\frac{\partial\zeta}{\partial\eta}\right)^2\right\}}\,d\xi\,d\eta,$$

where the integral is taken throughout the area bounded by the projection on the tangent plane at Q to the conducting surface of the curve of contact of the tangent cone from the point O to the conducting surface. Hence the value of the principal part of

$$\iint e^{-\iota\kappa(r_1+r)}\,dS$$

is the principal part of the integral

$$e^{-\iota\kappa(\mathrm{R}_1+\mathrm{R})}\iint e^{-\frac{1}{2}\iota\kappa(\xi^2\cos^2\phi+\eta^2)(\mathrm{R}_1{}^{-1}+\mathrm{R}^{-1})+\ldots}\sqrt{\{1+(\mathrm{A}\xi+\mathrm{H}\eta+\,.)^2+(\mathrm{H}\xi+\mathrm{B}\eta+\,.)^2\}}\,d\xi\,d\eta,$$

that is, writing

$$\kappa\xi = \xi', \qquad \kappa\eta = \eta',$$

it follows that

$$\mathrm{I} = \kappa^{-2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\iota\kappa(\mathrm{R}_1+\mathrm{R})-\frac{\iota}{2\kappa}(\xi'^2\cos^2\phi+\eta'^2)(\mathrm{R}_1{}^{-1}+\mathrm{R}^{-1})}\,d\xi'\,d\eta',$$

provided that the point Q, in which OP cuts the conducting surface, is not near to the curve of contact of the tangent cone from O to the surface with the surface. Evaluating the above integral, the value of I is given by

$$\mathrm{I} = \frac{2\pi}{\iota\kappa\cos\phi}\,\frac{\mathrm{R}\mathrm{R}_1}{\mathrm{R}+\mathrm{R}_1}\,e^{-\iota\kappa(\mathrm{R}_1+\mathrm{R})},$$

and therefore

$$\alpha_0 = -\frac{\iota\kappa^2}{\mathrm{V}}\,\frac{e^{\iota\kappa(\mathrm{V}t-\mathrm{R}_1-\mathrm{R})}}{(\mathrm{R}_1+\mathrm{R})\cos\phi}\,\{(l\xi_0+m\eta_0+n\zeta_0)(y-\eta_0)+n(\eta_0 z-\zeta_0 y)\}/\mathrm{R}\mathrm{R}_1,$$

$$\beta_0 = \frac{\iota\kappa^2}{\mathrm{V}}\,\frac{e^{\iota\kappa(\mathrm{V}t-\mathrm{R}_1-\mathrm{R})}}{(\mathrm{R}_1+\mathrm{R})\cos\phi}\,\{(l\xi_0+m\eta_0+n\zeta_0)(x-\xi_0)+n(\xi_0 z-\zeta_0 x)\}/\mathrm{R}\mathrm{R}_1,$$

$$\gamma_0 = \frac{\iota\kappa^2}{\mathrm{V}}\,\frac{e^{\iota\kappa(\mathrm{V}t-\mathrm{R}_1-\mathrm{R})}}{(\mathrm{R}_1+\mathrm{R})\cos\phi}\,\{n(\eta_0 x-\xi_0 y)\}/\mathrm{R}\mathrm{R}_1.$$

Now

$$\xi_0/\mathrm{R}_1 = (x-\xi_0)/\mathrm{R}, \qquad \eta_0/\mathrm{R}_1 = (y-\eta_0)/\mathrm{R}, \qquad \zeta_0/\mathrm{R}_1 = (z-\zeta_0)/\mathrm{R},$$

and

$$(l\xi_0+m\eta_0+n\zeta_0)/\mathrm{R}_1 = -\cos\phi,$$

whence, since

$$(y-\eta_0)/\mathrm{R} = y/(\mathrm{R}_1+\mathrm{R}), \qquad (x-\xi_0)/\mathrm{R} = x/(\mathrm{R}_1+\mathrm{R}),$$

$$\alpha_0 = \frac{\iota\kappa^2}{\mathrm{V}}\,\frac{y}{\mathrm{R}_1+\mathrm{R}}\,\frac{e^{\iota\kappa(\mathrm{V}t-\mathrm{R}_1-\mathrm{R})}}{\mathrm{R}_1+\mathrm{R}},$$

$$\beta_0 = -\frac{\iota\kappa^2}{\mathrm{V}}\,\frac{x}{\mathrm{R}_1+\mathrm{R}}\,\frac{e^{\iota\kappa(\mathrm{V}t-\mathrm{R}_1-\mathrm{R})}}{\mathrm{R}_1+\mathrm{R}},$$

$$\gamma_0 = 0.$$

Again, the components of the magnetic force at the point P due to the oscillator are

$$\alpha' = \frac{\kappa}{\mathrm{V}}\frac{\partial}{\partial y}\frac{e^{\iota\kappa(\mathrm{V}t-\mathrm{R}_1-\mathrm{R})}}{\mathrm{R}_1+\mathrm{R}}, \qquad \beta' = -\frac{\kappa}{\mathrm{V}}\frac{\partial}{\partial x}\frac{e^{\iota\kappa(\mathrm{V}t-\mathrm{R}_1-\mathrm{R})}}{\mathrm{R}_1+\mathrm{R}}, \qquad \gamma' = 0,$$

hence, to the order of approximation adopted, the components of the total magnetic force at the point P are given by

$$\alpha = \frac{\iota\kappa^2}{\mathrm{V}}\frac{y}{\mathrm{R}_1+\mathrm{R}}\frac{e^{\iota\kappa(\mathrm{V}t-\mathrm{R}_1-\mathrm{R})}}{\mathrm{R}_1+\mathrm{R}} - \frac{\iota\kappa^2}{\mathrm{V}}\frac{y}{\mathrm{R}_1+\mathrm{R}}\frac{e^{\iota\kappa(\mathrm{V}t-\mathrm{R}_1-\mathrm{R})}}{\mathrm{R}_1+\mathrm{R}} = 0,$$

$$\beta = -\frac{\iota\kappa^2}{\mathrm{V}}\frac{x}{\mathrm{R}_1+\mathrm{R}}\frac{e^{\iota\kappa(\mathrm{V}t-\mathrm{R}_1-\mathrm{R})}}{\mathrm{R}_1+\mathrm{R}} + \frac{\iota\kappa^2}{\mathrm{V}}\frac{x}{\mathrm{R}_1+\mathrm{R}}\frac{e^{\iota\kappa(\mathrm{V}t-\mathrm{R}_1-\mathrm{R})}}{\mathrm{R}_1+\mathrm{R}} = 0,$$

$$\gamma = 0;$$

that is, the principal part of the magnetic force vanishes at all points inside the tangent cone from the point O to the surface which are on the side of the surface remote from O and not near to the boundary of the tangent cone. The principal part of the electric force also vanishes for these points, and therefore the surface condition, that the electric force tangential to the surface vanishes, is satisfied for points on the portion of the conducting surface remote from O inside the tangent cone and not near to the curve of contact of the tangent cone with the surface.

When the point P is external to the tangent cone or inside the tangent cone on the same side of the conducting surface as the point O, the relations to be satisfied are

$$x_1/\mathrm{R}_1+x_2/\mathrm{R} = 0, \qquad z_1/\mathrm{R}_1-z_2/\mathrm{R} = 0,$$

and, as above, writing

$$x_1 = -\mathrm{R}_1\sin\phi, \qquad z_1 = -\mathrm{R}_1\cos\phi,$$

it follows that

$$x_2 = \mathrm{R}\sin\phi, \qquad y_2 = 0, \qquad z_2 = -\mathrm{R}\cos\phi,$$

and

$$r_1 = \mathrm{R}_1+\xi\sin\phi+\zeta\cos\phi+\tfrac{1}{2}(\xi^2\cos^2\phi+\eta^2)/\mathrm{R}_1+\ldots,$$

$$r = \mathrm{R}-\xi\sin\phi+\zeta\cos\phi+\tfrac{1}{2}(\xi^2\cos^2\phi+\eta^2)/\mathrm{R}\ldots,$$

whence

$$r_1+r = \mathrm{R}_1+\mathrm{R}+\tfrac{1}{2}(\xi^2\cos^2\phi+\eta^2)(\mathrm{R}_1^{-1}+\mathrm{R}^{-1})+2\zeta\cos\phi\ldots,$$

that is,

$$r_1+r = \mathrm{R}_1+\mathrm{R}+\tfrac{1}{2}\xi^2\{(\mathrm{R}_1^{-1}+\mathrm{R}^{-1})\cos^2\phi+2\mathrm{A}\cos\phi\}+2\mathrm{H}\cos\phi\,\xi\eta$$
$$+\tfrac{1}{2}\eta^2\{\mathrm{R}_1^{-1}+\mathrm{R}^{-1}+2\mathrm{B}\cos\phi\}+\ldots.$$

Writing

$$\xi = \xi_1\cos\theta-\eta_1\sin\theta, \qquad \eta = \xi_1\sin\theta+\eta_1\cos\theta,$$

and choosing $\theta$ so that the coefficient of $\xi_1$, $\eta_1$ vanishes, $\theta$ is given by

$$\cot 2\theta = \{2(\mathrm{A}-\mathrm{B})\cos\phi-(\mathrm{R}_1^{-1}+\mathrm{R}^{-1})\sin^2\phi\}/4\mathrm{H}\cos\phi,$$

and

$$r_1+r = \mathrm{R}_1+\mathrm{R}+\tfrac{1}{2}(\mathrm{A}_1\xi_1^2+\mathrm{B}_1\eta_1^2)+\ldots,$$

where

$$\mathrm{A}_1+\mathrm{B}_1 = (\mathrm{R}_1^{-1}+\mathrm{R}^{-1})(1+\cos^2\phi)+2(\mathrm{A}+\mathrm{B})\cos\phi,$$

$$\mathrm{A}_1\mathrm{B}_1 = (\mathrm{R}_1^{-1}+\mathrm{R}^{-1})^2\cos^2\phi+2(\mathrm{R}_1^{-1}+\mathrm{R}^{-1})(\mathrm{A}+\mathrm{B}\cos^2\phi)\cos\phi+4(\mathrm{AB}-\mathrm{H}^2)\cos^2\phi.$$

Now

$$\mathrm{A}_1\mathrm{B}_1 = \{(\mathrm{R}_1^{-1}+\mathrm{R}^{-1})\cos\phi+\mathrm{A}+\mathrm{B}\cos^2\phi\}^2-(\mathrm{A}-\mathrm{B}\cos^2\phi)^2-4\mathrm{H}^2\cos^2\phi,$$

and writing

$$\mathrm{A}-\mathrm{B}\cos^2\phi = 2\mathrm{H}\cos\phi\cot 2\psi,$$

this becomes

$$\mathrm{A}_1\mathrm{B}_1 = \cos^2\phi\,(\mathrm{R}^{-1}-f_1^{-1})\,(\mathrm{R}^{-1}-f_2^{-1}),$$

where

$$(\mathrm{R}_1^{-1}+f_1^{-1})\cos\phi+\mathrm{A}+\mathrm{B}\cos^2\phi-2\mathrm{H}\cos\phi\operatorname{cosec}2\psi = 0,$$

$$(\mathrm{R}_1^{-1}+f_2^{-1})\cos\phi+\mathrm{A}+\mathrm{B}\cos^2\phi+2\mathrm{H}\cos\phi\operatorname{cosec}2\psi = 0,$$

and

$$2\mathrm{H} = (f_1^{-1}-f_2^{-1})\sin\psi\cos\psi,$$

$$\mathrm{R}_1^{-1}+\cos^2\psi/f_1+\sin^2\psi/f_2+2\mathrm{B}\cos\phi = 0,$$

$$\mathrm{R}_1^{-1}+\sin^2\psi/f_1+\cos^2\psi/f_2+2\mathrm{A}\sec\phi = 0,$$

$$\cot 2\theta = \{(f_1^{-1}-\mathrm{R}^{-1})(\cos^2\psi-\sin^2\psi\cos^2\phi) + (f_2^{-1}-\mathrm{R}^{-1})(\sin^2\psi-\cos^2\psi\cos^2\phi)\}/(f_1^{-1}-f_2^{-1})\sin^2\psi\cos\phi.$$

The value of the principal part of the integral

$$\iint e^{-\iota\kappa(r_1+r)}\,d\mathrm{S}$$

is now given by

$$\mathrm{I} = \kappa^{-2}e^{-\iota\kappa(\mathrm{R}_1+\mathrm{R})}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{\iota}{2\kappa}(\mathrm{A}_1\xi'^2+\mathrm{B}_1\eta'^2)}\,d\xi'\,d\eta',$$

that is,

$$\mathrm{I} = \frac{2\pi}{\iota\kappa(\mathrm{A}_1\mathrm{B}_1)^{1/2}}\,e^{-\iota\kappa(\mathrm{R}_1+\mathrm{R})}$$

or

$$\mathrm{I} = \frac{2\pi}{\iota\kappa\cos\phi}(\mathrm{R}^{-1}-f_1^{-1})^{-1/2}(\mathrm{R}^{-1}-f_2^{-1})^{-1/2}\,e^{-\iota\kappa(\mathrm{R}_1+\mathrm{R})},$$

and therefore the principal parts of the components of the magnetic force at the point $(x, y, z)$ due to the surface distribution are given by

$$\alpha_0 = -\frac{\iota\kappa^2}{\mathrm{V}\cos\phi}(\mathrm{R}^{-1}-f_1^{-1})^{-1/2}(\mathrm{R}^{-1}-f_2^{-1})^{-1/2}\,e^{\iota\kappa(\mathrm{V}t-\mathrm{R}_1-\mathrm{R})} \{(l\xi_0+m\eta_0+n\zeta_0)(y-\eta_0)+n(\eta_0 z-\zeta_0 y)\}/\mathrm{R}^2\mathrm{R}_1^2,$$

$$\beta_0 = \frac{\iota\kappa^2}{\mathrm{V}\cos\phi}(\mathrm{R}^{-1}-f_1^{-1})^{-1/2}(\mathrm{R}^{-1}-f_2^{-1})^{-1/2}\,e^{\iota\kappa(\mathrm{V}t-\mathrm{R}_1-\mathrm{R})} \{(l\xi_0+m\eta_0+n\zeta_0)(x-\xi_0)+n(\xi_0 z-\zeta_0 x)\}/\mathrm{R}^2\mathrm{R}_1^2,$$

$$\gamma_0 = \frac{\iota\kappa^2}{\mathrm{V}\cos\phi}(\mathrm{R}^{-1}-f_1^{-1})^{-1/2}(\mathrm{R}^{-1}-f_2^{-1})^{-1/2}\,e^{\iota\kappa(\mathrm{V}t-\mathrm{R}_1-\mathrm{R})}\{n(\eta_0 x-\xi_0 y)\}/\mathrm{R}^2\mathrm{R}_1^2.$$

Writing

$$\xi_0/R_1 = \lambda_1, \qquad \eta_0/R_1 = \mu_1, \qquad \zeta_0/R_1 = \nu_1,$$

$$(x-\xi_0)/R = \lambda_2, \qquad (y-\eta_0)/R = \mu_2, \qquad (z-\zeta_0)/R = \nu_2,$$

it follows that

$$l\lambda_1+m\mu_1+n\nu_1 = -\cos\phi,$$

and the relations satisfied at the point Q are equivalent to

$$\lambda_2 = \lambda_1+2l\cos\phi, \qquad \mu_2 = \mu_1+2m\cos\phi, \qquad \nu_2 = \nu_1+2n\cos\phi,$$

whence the above values of $\alpha_0$, $\beta_0$, $\gamma_0$ may be written

$$\alpha_0 = \frac{\iota\kappa^2}{V}(R^{-1}-f_1^{-1})^{-1/2}(R^{-1}-f_2^{-1})^{-1/2}e^{\iota\kappa(Vt-R_1-R)}\{\mu_2+2n(m\nu_1-n\mu_1)\}/RR_1,$$

$$\beta_0 = \frac{\iota\kappa^2}{V}(R^{-1}-f_1^{-1})^{-1/2}(R^{-1}-f_2^{-1})^{-1/2}e^{\iota\kappa(Vt-R_1-R)}\{-\lambda_2+2n(n\lambda_1-l\nu_1)\}/RR_1,$$

$$\gamma_0 = \frac{\iota\kappa^2}{V}(R^{-1}-f_1^{-1})^{-1/2}(R^{-1}-f_2^{-1})^{-1/2}e^{\iota\kappa(Vt-R_1-R)}\{2n(l\mu_1-m\lambda_1)\}/RR_1.$$

The principal parts of the components of the electric force at the point $(x, y, z)$ due to the surface distribution are therefore given by

$$X_0 = -\iota\kappa^2(R^{-1}-f_1^{-1})^{-1/2}(R^{-1}-f_2^{-1})^{-1/2}e^{\iota\kappa(Vt-R_1-R)}\{\nu_1\lambda_1+2\nu_1 l\cos\phi+2nl\}/RR_1,$$

$$Y_0 = -\iota\kappa^2(R^{-1}-f_1^{-1})^{-1/2}(R^{-1}-f_2^{-1})^{-1/2}e^{\iota\kappa(Vt-R_1-R)}\{\mu_1\nu_1+2\nu_1 m\cos\phi+2mn\}/RR_1,$$

$$Z_0 = -\iota\kappa^2(R^{-1}-f_1^{-1})^{-1/2}(R^{-1}-f_2^{-1})^{-1/2}e^{\iota\kappa(Vt-R_1-R)}\{-\lambda_1^2-\mu_1^2+2\nu_1 n\cos\phi+2n^2\}/RR_1,$$

and, writing

$$x = \lambda r, \qquad y = \mu r, \qquad z = \nu r,$$

where

$$r^2 = x^2+y^2+z^2,$$

the principal parts of the components of the electric force at the point $(x, y, z)$ due to the oscillator are

$$\iota\kappa^2\nu\lambda\, e^{\iota\kappa(Vt-r)}/r, \qquad \iota\kappa^2\mu\nu\, e^{\iota\kappa(Vt-r)}/r, \qquad -\iota\kappa^2(\lambda^2+\mu^2)\, e^{\iota\kappa(Vt-r)}/r\,;$$

therefore the principal parts of the components of the total electric force at the point $(x, y, z)$ are given by

$$X = \iota\kappa^2\nu\lambda\, e^{\iota\kappa(Vt-r)}/r-\iota\kappa^2(R^{-1}-f_1^{-1})^{-1/2}(R^{-1}-f_2^{-1})^{-1/2}e^{\iota\kappa(Vt-R_1-R)} \\ \{\nu_1\lambda_1+2\nu_1 l\cos\phi+2nl\}/RR_1,$$

$$Y = \iota\kappa^2\mu\nu\, e^{\iota\kappa(Vt-r)}/r-\iota\kappa^2(R^{-1}-f_1^{-1})^{-1/2}(R^{-1}-f_2^{-1})^{-1/2}e^{\iota\kappa(Vt-R_1-R)} \\ \{\mu_1\nu_1+2\nu_1 m\cos\phi+2mn\}/RR_1,$$

$$Z = -\iota\kappa^2(\lambda^2+\mu^2)\, e^{\iota\kappa(Vt-r)}/r-\iota\kappa^2(R^{-1}-f_1^{-1})^{-1/2}(R^{-1}-f_2^{-1})^{-1/2}e^{\iota\kappa(Vt-R_1-R)} \\ \{-\lambda_1^2-\mu_1^2+2\nu_1 n\cos\phi+2n^2\}/RR_1.$$

When the point P is on the surface of the conductor,

$$\mathrm{R} = 0, \qquad r = \mathrm{R}_1, \qquad \lambda = \lambda_1, \qquad \mu = \mu_1, \qquad \nu = \nu_1,$$

and therefore the principal parts of the components of the electric force at a point on the surface are given by

$$\mathrm{X} = -2\iota\kappa^2 l\,(\nu_1 \cos\phi + n)\, e^{\iota\kappa(\mathrm{V}t - \mathrm{R}_1)}/\mathrm{R}_1,$$

$$\mathrm{Y} = -2\iota\kappa^2 m\,(\nu_1 \cos\phi + n)\, e^{\iota\kappa(\mathrm{V}t - \mathrm{R}_1)}/\mathrm{R}_1,$$

$$\mathrm{Z} = -2\iota\kappa^2 n\,(\nu_1 \cos\phi + n)\, e^{\iota\kappa(\mathrm{V}t - \mathrm{R}_1)}/\mathrm{R}_1,$$

hence the principal part of the electric force at a point on the surface is normal to the surface, and therefore the principal part of the electric force satisfies the condition that the electric force tangential to the surface vanishes. It follows from this that the principal parts of the components of the magnetic and electric forces at points not near to the boundary of the geometrical shadow are those given above.

To find the region within which the point P lies when the above values cease to represent the principal parts of the components of the magnetic and electric forces due to the obstacle, it is necessary to find the order of the terms neglected by taking the limits of the integral representing the principal part of the integral

$$\iint e^{-\iota\kappa(r_1 + r)}\, d\mathrm{S}$$

to be infinite. In the evaluation of I this integral was replaced by an integral taken throughout the area enclosed by the curve which is the projection on the tangent plane at the point Q of the curve of contact of the tangent cone from the point O to the surface of the obstacle with this surface. The actual limits of the integral

$$\iint e^{-\frac{\iota}{2\kappa}(\xi'^2 \cos^2\phi + \eta'^2)(\mathrm{R}_1{}^{-1} + \mathrm{R}^{-1})}\, d\xi'\, d\eta'$$

are quantities of the order $\kappa d$ or quantities of higher order, where $d$ is the least distance of the point Q from the boundary of the curve throughout whose area the integration is taken; hence the part of the integral neglected by taking the limits to be infinite is at most of the order $(\kappa d^2/\mathrm{R}_1 + \kappa d^2/\mathrm{R})^{-1/2}$ compared with the part retained, when the point P is inside the tangent cone from the point O to the surface of the obstacle and on the side of the surface remote from the point O. Again, when the point P is outside the tangent cone from the point O to the surface, the actual limits of the corresponding integral are also quantities of the order $\kappa d$ or quantities of a higher order, and therefore, since $\mathrm{A} + \mathrm{B}$ and $\mathrm{AB} - \mathrm{H}^2$ are both positive, the part of the integral neglected by taking the limits to be infinite is at most of the order $(\kappa d^2/\mathrm{R}_1 + \kappa d^2/\mathrm{R})^{-1/2}$ compared with the part retained. The region within which the point P lies when the values obtained above for the principal parts of the magnetic and electric forces cease to represent them is determined by the condition that the point Q related to the point P in the manner already defined is so close to the curve of contact of the tangent cone from the point O to the surface of the obstacle with the surface that the quantity $(\kappa d^2/\mathrm{R}_1+\kappa d^2/\mathrm{R})^{-1/2}$ is a quantity of the order of unity or of a higher order.

When the point P is sufficiently distant from the surface of the tangent cone from the point O to the surface of the obstacle, the limits of the integral may be taken to be infinite, and the order of the difference between the components of the actual magnetic and electric forces due to the obstacle and the components of the magnetic and electric forces due to the assumed electric current distribution may be obtained as follows. In the exponent of $e^{-\iota\kappa(r_1+r)}$ the terms up to and including those of the fourth degree in $\xi$ and $\eta$ must be retained, and in the other factors of the integrands the terms up to and including the terms of the second degree in $\xi$ and $\eta$ must be retained; this requires that in the equation to the surface

$$\zeta = \tfrac{1}{2}(\mathrm{A}\xi^2+2\mathrm{H}\xi\eta+\mathrm{B}\eta^2)+\ldots+\ldots,$$

terms up to and including the terms of the fourth degree in $\xi$ and $\eta$ are retained, and the resulting integrals are now of the form

$$\iint(g_0+g_1+g_2+\kappa g_3+\kappa g_4)\, e^{-1/2\iota\kappa(\mathrm{A}_1\xi^2+\mathrm{B}_1\eta^2)}\, d\xi\, d\eta,$$

where $g_n$ denotes a homogeneous function of $\xi$, $\eta$ of degree $n$. The terms of odd degree in $\xi$, $\eta$ contribute zero to the result and the integral is equal to

$$\iint(g_0+g_2+\kappa g_4)\, e^{-1/2\iota\kappa(\mathrm{A}_1\xi^2+\mathrm{B}_1\eta^2)}\, d\xi\, d\eta.$$

Now, if the conducting surface were the infinite plane coinciding with the tangent plane at the point Q to the surface of the obstacle, the corresponding integral would be

$$\iint(g_0+g'_2+\kappa g'_4)\, e^{-1/2\iota\kappa(\mathrm{A}'_1\xi^2+\mathrm{B}'_1\eta^2)}\, d\xi\, d\eta,$$

where this integral is equal to the value of the preceding integral when the quantities A, B, H, &c., in the equation to the surface are made zero. Further, for the surface $\zeta = 0$, the condition that the tangential electric force due to the assumed electric current distribution vanishes at the surface is accurately satisfied, and therefore the principal part of the electric force tangential to the surface of the obstacle at the point Q is the value at Q of the principal part of an integral of the form

$$\iint\left[(g_0+g_2+\kappa g_4)\, e^{-1/2\iota\kappa(\mathrm{A}_1\xi^2+\mathrm{B}_1\eta^2)}-(g_0+g'_2+\kappa g'_4)\right] e^{-1/2\iota\kappa(\mathrm{A}'_1\xi^2+\mathrm{B}'_1\eta^2)}\, d\xi\, d\eta,$$

which vanishes when A, B, H, &c., vanish, and hence is at most of the order $(\kappa\rho)^{-1}$, where $\rho$ is the least radius of curvature of the surface at the point Q, it being assumed that the values of $\xi\frac{\partial^3\zeta}{\partial\xi^3}$, $\xi^2\frac{\partial^4\zeta}{\partial\xi^4}$, and the similar quantities in the immediate neighbourhood of the point Q are not of an order higher than $\rho^{-1}$. Therefore the difference between the components of the actual magnetic and electric forces due to the obstacle at the point P, not near to the surface of the tangent cone from the point O to the surface of the obstacle, and the components of the magnetic and electric forces due to the assumed electric current distribution on the surface of the obstacle is at most of the order $(\kappa\rho)^{-1}$, where $\rho$ is the least radius of curvature of the surface at the point Q, which is related to the point P in the manner already defined.*

It has been shown above that the principal parts of the components of the magnetic force due to the assumed electric current distribution on the surface of the obstacle are

$$\frac{\iota\kappa^2}{\mathrm{V}}(\mathrm{R}^{-1}-f_1^{-1})^{-1/2}(\mathrm{R}^{-1}-f_2^{-1})^{-1/2}e^{\iota\kappa(\mathrm{V}t-\mathrm{R}_1-\mathrm{R})}\{\mu_2+2n(m\nu_1-n\mu_1)\}/\mathrm{RR}_1,$$

$$\frac{\iota\kappa^2}{\mathrm{V}}(\mathrm{R}^{-1}-f_1^{-1})^{-1/2}(\mathrm{R}^{-1}-f_2^{-1})^{-1/2}e^{\iota\kappa(\mathrm{V}t-\mathrm{R}_1-\mathrm{R})}\{-\lambda_2+2n(n\lambda_1-l\nu_1)\}/\mathrm{RR}_1,$$

$$\frac{\iota\kappa^2}{\mathrm{V}}(\mathrm{R}^{-1}-f_1^{-1})^{-1/2}(\mathrm{R}^{-1}-f_2^{-1})^{-1/2}e^{\iota\kappa(\mathrm{V}t-\mathrm{R}_1-\mathrm{R})}\{2n(l\mu_1-m\lambda_1)\}/\mathrm{RR}_1\,;$$

in these expressions the quantities $f_1$, $f_2$ are both negative, since A+B and $\mathrm{AB}-\mathrm{H}^2$ are both positive, hence when the obstacle is a convex solid the principal parts of the components of the magnetic force in the reflected waves are given at all points P not near to the surface of the tangent cone from the point O to the surface of the obstacle by the above expressions which are everywhere finite. The points for which $\mathrm{R}-f_1$, $\mathrm{R}-f_2$ vanish are situated on the line PQ produced and determine the positions of virtual focal lines; the directions of these focal lines are given by

$$\tan\theta = \tan\psi\sec\phi, \qquad \text{when} \qquad \mathrm{R} = f_1,$$

and by

$$\tan\theta = \tan\psi\cos\phi, \qquad \text{when} \qquad \mathrm{R} = f_2.$$

If the obstacle is not a convex solid, provided that there is no point P on its surface such that a prolate spheroid whose foci are the points O and P can be drawn to touch the surface of the obstacle, the same analysis applies, but, P being now a point not on the surface of the obstacle, if the surface is concave towards O and P at the point Q $(\xi_0, \eta_0, \zeta_0)$ determined as before, the points $P_1$, $P_2$ on the straight line QP for which $R-f_1$ or $R-f_2$ vanish are both external to the surface of the obstacle if

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* The value of the principal parts of these differences can be obtained by placing on the surface of the obstacle an electric current distribution of order $(\kappa\rho)^{-1}$ which will balance the unbalanced tangential electric force of this order.

$$A+B\cos^2\phi+\cos\phi/R_1$$

is negative, for then $f_1$ and $f_2$ are both positive. The principal parts of the magnetic force in the reflected waves given by the above expressions tend towards infinite values as either of the points $P_1$, $P_2$ is approached, and the points $P_1$, $P_2$ now determine the positions of actual focal lines whose directions are given by

$$\tan\theta = \tan\psi\sec\phi, \qquad R = f_1,$$
$$\tan\theta = \tan\psi\cos\phi, \qquad R = f_2.$$

The principal parts of the components of the magnetic force were evaluated above on the assumption that $R^{-1}-f_1^{-1}$, $R^{-1}-f_2^{-1}$ were both finite; if either of these quantities is very small, some of the terms neglected in that evaluation are of the same order as those retained, and it is necessary to calculate the principal part of the magnetic force taking account of these terms. In the equation of the surface in the neighbourhood of the point Q $(\xi_0, \eta_0, \zeta_0)$ the terms of the third order must be retained and the equation to the surface with Q as origin of co-ordinates and the same axes of reference as formerly is now

$$\zeta = \tfrac{1}{2}(A\xi^2+2H\xi\eta+B\eta^2)+\tfrac{1}{6}(C\xi^3+3L\xi^2\eta+3M\xi\eta^2+D\eta^3).$$

The value of $r_1+r$ is now given by

$$r_1+r = R_1+R+\tfrac{1}{2}(A'\xi^2+2H'\xi\eta+B'\eta^2)+\tfrac{1}{6}(C'\xi^3+3L'\xi^2\eta+3M'\xi\eta^2+D'\eta^3)$$

where

$$A' = (R_1^{-1}+R^{-1})\cos^2\phi+2A\cos\phi, \quad B' = (R_1^{-1}+R^{-1})+2B\cos\phi, \quad H' = 2H\cos\phi,$$
$$C' = 2C\cos\phi+3A\sin\phi\cos\phi\,(R^{-1}-R_1^{-1})+3\sin\phi\cos^2\phi\,(1/R^2-1/R_1^2),$$
$$L' = 2L\cos\phi+2H\sin\phi\cos\phi\,(R^{-1}-R_1^{-1}),$$
$$M' = 2M\cos\phi+B\sin\phi\cos\phi\,(R^{-1}-R_1^{-1})+\sin\phi\,(1/R^2-1/R_1^2),$$
$$D' = 2D\cos\phi.$$

At the point $P_1$, for which $R = f_1$, these relations become

$$A' = K\cos^2\phi\cos^2\psi, \qquad B' = K\sin^2\psi, \qquad H' = K\sin\psi\cos\psi\cos\phi, \text{ \&c.},$$

where

$$f_1^{-1}-f_2^{-1} = K,$$

and, turning the axes of $\xi$ and $\eta$ through the angle $\theta$ given by $\tan\theta = \tan\psi\sec\phi$, $r_1+r$ is given by

$$r_1+r = R_1+R+\tfrac{1}{2}(A_1\xi^2+B_1\eta^2)+\tfrac{1}{6}(C_1\xi^3+3L_1\xi^2\eta+3M_1\xi\eta^2+D_1\eta^3),$$

where

$$A_1 = K(\cos^2\phi\cos^2\psi+\sin^2\psi), \qquad B_1 = 0,$$
$$D_1 = [2\cos\phi(-C\sin^3\psi+3L\sin^2\psi\cos\phi\cos\psi+-3M\sin\psi\cos^2\phi\cos^2\psi+D\cos^3\phi\cos^3\psi)$$
$$+\tfrac{3}{2}(1/R_1^2-1/R^2)\sin\psi\sin\phi\cos^2\phi]\times(\cos^2\phi\cos^2\psi+\sin^2\psi)^{-3/2},$$

and the principal part of the integral

$$\iint e^{-\iota\kappa(r_1+r)}\,dS$$

is equal to the principal part of the integral

$$e^{-\iota\kappa(R_1+R)}\iint e^{-1/2\,\iota\kappa A_1\xi^2-1/6\,\iota\kappa(C_1\xi^3+3L_1\xi^2\eta+3M_1\xi\eta^2+D_1\eta^3)}\,d\xi\,d\eta,$$

that is to the integral

$$\kappa^{-2}e^{-\iota\kappa(R_1+R)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\frac{\iota}{\kappa}A_1\xi_1^2-\frac{\iota}{6\kappa^2}D_1\eta_1^3}\,d\xi_1\,d\eta_1,$$

which is equal to

$$\kappa^{-2}e^{-\iota\kappa(R_1+R)}\,\pi^{1/2}\left(\frac{\iota}{2\kappa}A_1\right)^{-1/2}\int_{-\infty}^{\infty}e^{-\frac{\iota}{6\kappa^2}D_1\eta_1^3}\,d\eta_1,$$

that is to

$$\pi^{1/2}\kappa^{-2}e^{-\iota\kappa(R_1+R)}\left(\frac{\iota}{2\kappa}A_1\right)^{-1/2}\left[\left(\frac{\iota D_1}{6\kappa^2}\right)^{-1/3}+\left(\frac{-\iota D_1}{6\kappa^2}\right)^{-1/3}\right]\tfrac{1}{3}\Gamma(\tfrac{1}{3}),$$

whence

$$I = \pi^{1/2}\kappa^{-5/6}2^{5/6}3^{-1/6}\Gamma(\tfrac{1}{3})\,e^{-\iota\kappa(R_1+R)+\frac{\pi\iota}{4}}A_1^{-1/2}D_1^{-1/3},$$

where $A_1$ and $D_1$ have the values given above. Therefore at the point $P_1$ the principal parts of the components of the magnetic force in the reflected waves are given by

$$\alpha_0 = -\frac{\kappa^{13/6}}{\pi V}6^{-1/6}\Gamma(\tfrac{1}{2})\Gamma(\tfrac{1}{3})A_1^{-1/2}D_1^{-1/3}e^{\iota\kappa(Vt-R_1-R)+\frac{\pi\iota}{4}}\cos\phi\,[\mu_2+2n(m\nu_1-n\mu_1)]/R_1R,$$

$$\beta_0 = -\frac{\kappa^{13/6}}{\pi V}6^{-1/6}\Gamma(\tfrac{1}{2})\Gamma(\tfrac{1}{3})A_1^{-1/2}D_1^{-1/3}e^{\iota\kappa(Vt-R_1-R)+\frac{\pi\iota}{4}}\cos\phi\,[-\lambda_2+2n(n\lambda_1-l\nu_1)]/R_1R,$$

$$\gamma_0 = -\frac{\kappa^{13/6}}{\pi V}6^{-1/6}\Gamma(\tfrac{1}{2})\Gamma(\tfrac{1}{3})A_1^{-1/2}D_1^{-1/3}e^{\iota\kappa(Vt-R_1-R)+\frac{\pi\iota}{4}}\cos\phi\,[2n(l\mu_1-m\lambda_1)]/R_1R.$$

Similarly, at the point $P_2$, for which $R = f_2$, the principal parts of the components of the magnetic force are given by

$$\alpha_0 = -\frac{\kappa^{13/6}}{\pi V}6^{-1/6}\Gamma(\tfrac{1}{2})\Gamma(\tfrac{1}{3})B_1^{-1/2}C_1^{-1/3}e^{\iota\kappa(Vt-R_1-R)+\frac{\pi\iota}{4}}\cos\phi\,[\mu_2+2n(m\nu_1-n\mu_1)]/R_1R,$$

$$\beta_0 = -\frac{\kappa^{13/6}}{\pi V}6^{-1/6}\Gamma(\tfrac{1}{2})\Gamma(\tfrac{1}{3})B_1^{-1/2}C_1^{-1/3}e^{\iota\kappa(Vt-R_1-R)+\frac{\pi\iota}{4}}\cos\phi\,[-\lambda_2+2n(n\lambda_1-l\nu_1)]/R_1R,$$

$$\gamma_0 = -\frac{\kappa^{13/6}}{\pi V}6^{-1/6}\Gamma(\tfrac{1}{2})\Gamma(\tfrac{1}{3})B_1^{-1/2}C_1^{-1/3}e^{\iota\kappa(Vt-R_1-R)+\frac{\pi\iota}{4}}\cos\phi\,[2n(l\mu_1-m\lambda_1)]/R_1R,$$

where

$$\mathrm{B}_1 = -\mathrm{K}\,(\sin^2\psi\cos^2\phi+\cos^2\psi),$$

$$\begin{aligned}\mathrm{C}_1 = [&2\cos\phi\,(\mathrm{C}\cos^3\psi+3\mathrm{L}\sin\psi\cos^2\psi\cos\phi+3\mathrm{M}\sin^2\psi\cos\psi\cos^2\phi+\mathrm{D}\sin^3\psi\cos^3\phi)\\ &+\tfrac{3}{2}\,(1/\mathrm{R}^2-1/\mathrm{R}_1^{\,2})\sin\phi\cos^2\phi\cos\psi]\,[\cos^2\psi+\sin^2\psi\cos^2\phi]^{-3/2}.\end{aligned}$$

From these results it follows that the intensity of the reflected waves at a point on a caustic is of a higher order than at an ordinary point, the ratio of the two intensities depending on $\lambda^{-1/3}$, where $\lambda$ is the wave-length of the waves. In the above investigation it has been assumed that, when $\mathrm{R} = f_1$, $\mathrm{D}_1$ and $\mathrm{A}_1$ are finite, and that, when $\mathrm{R} = f_2$, $\mathrm{C}_1$ and $\mathrm{B}_1$ are finite; if, when $\mathrm{R} = f_1$, $\mathrm{A}_1$ is finite and $\mathrm{D}_1$ vanishes, terms of the fourth order must be retained and the ratio of the intensity at such a point on a caustic to the intensity at an ordinary point depends on $\lambda^{-1/2}$. Similarly, if, when $\mathrm{R} = f_2$, $\mathrm{B}_1$ is finite and $\mathrm{C}_1$ vanishes, the ratio of the intensity at the corresponding point on the caustic to the intensity at an ordinary point depends on $\lambda^{-1/2}$. Further, at a point on the intersection of the two sheets of the caustic $\mathrm{A}_1$ and $\mathrm{B}_1$ vanish simultaneously and, if $\mathrm{C}_1$ and $\mathrm{D}_1$ are both finite, the ratio of the intensity at this point to the intensity at an ordinary point depends on $\lambda^{-2/3}$. The other cases which depend on the vanishing of more of the constants $\mathrm{A}_1$, $\mathrm{B}_1$, $\mathrm{C}_1$, &c., can be similarly treated.

At a point in the neighbourhood of the caustic either $\mathrm{R}-f_1$ or $\mathrm{R}-f_2$ is a small quantity; if $\mathrm{R}-f_1$ is small and $\mathrm{R}-f_2$ is not small, $\mathrm{B}_1$ is a small quantity and the value of I the principal part of the integral

$$\iint e^{-\iota\kappa\,(r_1+r)}\,d\mathrm{S}$$

is given by

$$\mathrm{I} = \kappa^{-2}\,e^{-\iota\kappa\,(\mathrm{R}_1+\mathrm{R})}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{\iota}{2\kappa}(\mathrm{A}_1\xi^2+\mathrm{B}_1\eta^2)-\frac{\iota}{6\kappa^2}\mathrm{D}_1\eta^3}\,d\xi\,d\eta,$$

that is

$$\mathrm{I} = \kappa^{-2}\pi^{1/2}\left(\frac{\iota}{2\kappa}\,\mathrm{A}_1\right)^{-1/2}\int_{-\infty}^{\infty} e^{-\frac{\iota}{2\kappa}\mathrm{B}_1\eta^2-\frac{\iota}{6\kappa^2}\mathrm{D}_1\eta^3}\,d\eta\,e^{-\iota\kappa\,(\mathrm{R}_1+\mathrm{R})},$$

or

$$\mathrm{I} = 2\pi^{1/2}\kappa^{-2}\left(\frac{\iota}{2\kappa}\,\mathrm{A}_1\right)^{-1/2}\left(\frac{3\pi\kappa^2}{\mathrm{D}_1}\right)^{1/3}\mathrm{W}\,e^{-\iota\kappa\,(\mathrm{R}_1+\mathrm{R})},$$

where

$$\mathrm{W} = \int_0^{\infty}\cos\tfrac{1}{2}\pi\,(\zeta^3-\mu\zeta)\,d\zeta, \qquad \text{and} \qquad \mu = 12^{1/3}\mathrm{B}_1^{\,2}\lambda^{-2/3}\mathrm{D}_1^{\,-4/3}.$$

Therefore at a point in the neighbourhood of a caustic the principal parts of the components of the magnetic force in the reflected waves are given by

$$\alpha_0 = -2^{1/2}3^{1/3}\pi^{-1/6}\mathrm{V}^{-1}\kappa^{13/6}\mathrm{A}_1^{\,-1/2}\mathrm{D}_1^{\,-1/3}\mathrm{W}e^{\iota\kappa\,(\mathrm{V}t-\mathrm{R}_1-\mathrm{R})-\frac{\pi\iota}{4}}\cos\phi\,[\mu_2+2n\,(m\nu_1-n\mu_1)]/\mathrm{R}_1\mathrm{R},$$

$$\beta_0 = -2^{1/2}3^{1/3}\pi^{-1/6}\mathrm{V}^{-1}\kappa^{13/6}\mathrm{A}_1^{\,-1/2}\mathrm{D}_1^{\,-1/3}\mathrm{W}e^{\iota\kappa\,(\mathrm{V}t-\mathrm{R}_1-\mathrm{R})-\frac{\pi\iota}{4}}\cos\phi\,[-\lambda_2+2n\,(n\lambda_1-l\nu_1)]/\mathrm{R}_1\mathrm{R},$$

$$\gamma_0 = -2^{1/2}3^{1/3}\pi^{-1/6}\mathrm{V}^{-1}\kappa^{13/6}\mathrm{A}_1^{\,-1/2}\mathrm{D}_1^{\,-1/3}\mathrm{W}e^{\iota\kappa\,(\mathrm{V}t-\mathrm{R}_1-\mathrm{R})-\frac{\pi\iota}{4}}\cos\phi\,[2n\,(l\mu_1-m\lambda_1)]/\mathrm{R}_1\mathrm{R},$$

where W has the value given above. When $R - f_2$ is small and $R - f_1$ is not small, the principal parts of the components of the magnetic force in the reflected waves are given by the corresponding expressions in which $A_1$ is replaced by $B_1$, $B_1$ is replaced by $A_1$, and $D_1$ is replaced by $C_1$. At a point in the neighbourhood of the intersection of the two sheets of the caustic, $A_1$ and $B_1$ are both small, and the expressions for the principal parts of the components of the magnetic force in the reflected waves will contain the product of two integrals which are values of W corresponding to two values of $\mu$.

When the obstacle is of any form, if P is a point not inside the boundary of the geometrical shadow, there may be more than one point Q on the surface of the obstacle such that $OQ + QP$ is stationary, or there may be points $Q$, $Q_1$, $Q_2$, … $Q_n$ such that $OQ + QQ_1 + Q_1Q_2 + \ldots Q_nP$ is stationary; in the first of these cases the principal parts of the components of the magnetic force in the reflected waves is the same for all the different points Q; in the second case the principal parts of the magnetic force in the reflected waves at P is obtained by placing an electric current distribution on the surface in the neighbourhood of Q double that due to the tangential electric force from O, an electric current distribution in the neighbourhood of $Q_1$ double that due to the tangential electric force arising from the current distribution in the neighbourhood of Q and so on.

The magnetic and electric forces due to the assumed distribution over the surface of the obstacle have still to be obtained at points which are near to the boundary of the geometrical shadow. Taking first the case where the point P is inside the tangent cone from the point O to the surface of the obstacle, and on the side of the obstacle remote from the point O, the principal parts of $\alpha$, $\beta$, $\gamma$, where

$$\alpha = -\frac{\kappa}{2\pi V}\iint\left[\frac{y-\eta}{r}\,\frac{l\xi+m\eta+n\zeta}{r_1}+\frac{n\,(z\eta-y\zeta)}{rr_1}\right]\frac{\partial^2}{\partial r\,\partial r_1}\,\frac{e^{\iota\kappa(Vt-r_1-r)}}{rr_1}\,dS,$$

$$\beta = \frac{\kappa}{2\pi V}\iint\left[\frac{x-\xi}{r}\,\frac{l\xi+m\eta+n\zeta}{r_1}+\frac{n\,(z\xi-x\zeta)}{rr_1}\right]\frac{\partial^2}{\partial r\,\partial r_1}\,\frac{e^{\iota\kappa(Vt-r_1-r)}}{rr_1}\,dS,$$

$$\gamma = \frac{\kappa}{2\pi V}\iint\frac{n\,(x\eta-y\xi)}{rr_1}\,\frac{\partial^2}{\partial r\,\partial r_1}\,\frac{e^{\iota\kappa(Vt-r_1-r)}}{rr_1}\,dS,$$

have to be calculated when the limits can no longer be taken infinite for the purpose. As before, the principal parts of $\alpha$, $\beta$, $\gamma$ are given by

$$\alpha_0 = \frac{\kappa^3}{2\pi V}\,\frac{y-\eta_0}{R}\,I, \quad \beta_0 = -\frac{\kappa^3}{2\pi V}\,\frac{x-\xi_0}{R}\,I, \quad \gamma_0 = 0,$$

where I is the principal part of the integral

$$\iint\frac{l\xi_0+m\eta_0+n\zeta_0}{R}\,\frac{e^{\iota\kappa(Vt-r_1-r)}}{R_1R}\,dS.$$

Now

$$\iint \frac{l\xi_0+m\eta_0+n\zeta_0}{\mathrm{R}} \frac{e^{\iota\kappa(\mathrm{V}t-r_1-r)}}{\mathrm{R}_1\mathrm{R}}\, d\mathrm{S} = -\iint \frac{e^{\iota\kappa(\mathrm{V}t-r_1-r)}}{\mathrm{R}_1\mathrm{R}}\, d\sigma,$$

where $d\sigma$ is the projection of the element of area $d\mathrm{S}$ on a plane perpendicular to the straight line OP; to evaluate this integral let the axes of reference be OP the axis of $z$, the perpendicular to OP in the plane of incidence the axis of $x$, and the straight line perpendicular to these two the axis of $y$. Let $\xi$, $\eta$, $\zeta$ now denote the co-ordinates of a point on the surface, and $\xi_1$, $0_1$, $\zeta_1$ the co-ordinates of the point of contact R of the tangent from O to the surface in the plane of incidence; the co-ordinates of the point P are 0, 0, $\mathrm{R}_1+\mathrm{R}$, then

$$\iint e^{-\iota\kappa(r_1+r)}\, d\sigma = \int^{\xi_1}\!\!\int e^{-\iota\kappa(r_1+r)}\, d\xi\, d\eta,$$

that is, the principal part of this integral is given by

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\iota\kappa(r_1+r)}\, d\xi\, d\eta - \int_{\xi_1}\!\!\int e^{-\iota\kappa(r_1+r)}\, d\xi\, d\eta.$$

It has already been proved that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\iota\kappa(r_1+r)}\, d\xi\, d\eta = \pi\left[\frac{\iota\kappa}{2}\left(\frac{1}{\mathrm{R}}+\frac{1}{\mathrm{R}_1}\right)\right]^{-1} e^{-\iota\kappa(\mathrm{R}_1+\mathrm{R})};$$

to obtain the value of the second integral

$$r_1^2 = \xi^2+\eta^2+\zeta^2, \qquad r^2 = \xi^2+\eta^2+(\mathrm{R}_1+\mathrm{R}-\zeta)^2,$$

and observing that $\xi_1$, $\zeta_1$ correspond to the lower limit of the integral

$$r_1 = \zeta_1+\frac{1}{2\zeta_1}(\xi^2+\eta^2) + \&c.,$$

$$r = \mathrm{R}_1+\mathrm{R}-\zeta_1+\frac{1}{2(\mathrm{R}_1+\mathrm{R}-\zeta_1)}(\xi^2+\eta^2) + \&c.,$$

therefore

$$\int_{\xi_1}\!\!\int e^{-\iota\kappa(r_1+r)}\, d\sigma = e^{-\iota\kappa(\mathrm{R}_1+\mathrm{R})}\int_{\xi_1}\!\!\int e^{-\frac{\iota\kappa}{2}\left(\frac{1}{\zeta_1}+\frac{1}{\mathrm{R}_1+\mathrm{R}-\zeta_1}\right)(\xi^2+\eta^2)}\, d\xi\, d\eta;$$

hence, unless the radius of curvature of the projection of the curve of contact of the tangent cone with the surface on the plane perpendicular to OP is a small quantity of the order of $\xi_1$, the principal part of the above integral is given by

$$\kappa^{-2}e^{-\iota\kappa(\mathrm{R}_1+\mathrm{R})}\int_{\kappa\xi_1}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{\iota}{2\kappa}\left(\frac{1}{\zeta_1}+\frac{1}{\mathrm{R}_1+\mathrm{R}-\zeta_1}\right)(u^2+v^2)}\, du\, dv,$$

that is, the principal part of

$$\int_{\xi_1}\!\!\int e^{-\iota\kappa(r_1+r)}\, d\sigma$$

is given by

$$\pi\kappa^{-1}\iota^{-1/2}2^{1/2}e^{-\iota\kappa(\mathrm{R}_1+\mathrm{R})}\frac{\mathrm{R}_1\mathrm{R}}{\mathrm{R}_1+\mathrm{R}}\int_{u_0}^{\infty}e^{-1/2\pi\iota u^2}\,du,$$

where

$$u_0=\left\{\frac{\kappa}{\pi}\left(\frac{1}{\zeta_1}+\frac{1}{\mathrm{R}_1+\mathrm{R}-\zeta_1}\right)\right\}^{1/2}\xi_1.$$

Therefore

$$\mathrm{I}=-\frac{2\pi}{\iota\kappa}\frac{\mathrm{R}_1\mathrm{R}}{\mathrm{R}_1+\mathrm{R}}e^{\iota\kappa(\mathrm{V}t-\mathrm{R}_1-\mathrm{R})}\left\{1-2^{-1/2}\iota^{1/2}\int_{u_0}^{\infty}e^{-1/2\pi\iota u^2}\,du\right\},$$

whence, writing

$$2^{-1/2}e^{\frac{\pi\iota}{4}}\int_{u_0}^{\infty}e^{-1/2\pi\iota u^2\,du}=\mathrm{L},$$

it follows that the principal parts of the components of the magnetic force due to the assumed surface distribution are given by

$$\alpha_0=\frac{\iota\kappa^2}{\mathrm{V}}\frac{e^{\iota\kappa(\mathrm{V}t-\mathrm{R}_1-\mathrm{R})}}{\mathrm{R}_1+\mathrm{R}}\frac{y-\eta_0}{\mathrm{R}}(1-\mathrm{L}),$$

$$\beta_0=-\frac{\iota\kappa^2}{\mathrm{V}}\frac{e^{\iota\kappa(\mathrm{V}t-\mathrm{R}_1-\mathrm{R})}}{\mathrm{R}_1+\mathrm{R}}\frac{x-\xi_0}{\mathrm{R}}(1-\mathrm{L}),$$

$$\gamma_0=0.$$

Now, $\alpha'$, $\beta'$, $\gamma'$, the principal parts of the components of the magnetic force due to the oscillator at the point O are given by

$$\alpha'=-\frac{\iota\kappa^2y}{\mathrm{V}(\mathrm{R}_1+\mathrm{R})^2}e^{\iota\kappa(\mathrm{V}t-\mathrm{R}_1-\mathrm{R})}=-\frac{\iota\kappa^2}{\mathrm{V}}\frac{y-\eta_0}{\mathrm{R}}e^{\iota\kappa(\mathrm{V}t-\mathrm{R}_1-\mathrm{R})}/(\mathrm{R}_1+\mathrm{R}),$$

$$\beta'=\frac{\iota\kappa^2x}{\mathrm{V}(\mathrm{R}_1+\mathrm{R})^2}e^{\iota\kappa(\mathrm{V}t-\mathrm{R}_1-\mathrm{R})}=\frac{\iota\kappa^2}{\mathrm{V}}\frac{x-\xi_0}{\mathrm{R}}e^{\iota\kappa(\mathrm{V}t-\mathrm{R}_1-\mathrm{R})}/(\mathrm{R}_1+\mathrm{R}),$$

$$\gamma'=0,$$

for

$$\frac{x}{\mathrm{R}_1+\mathrm{R}}=\frac{x-\xi_0}{\mathrm{R}},\qquad\frac{y}{\mathrm{R}_1+\mathrm{R}}=\frac{y-\eta_0}{\mathrm{R}};$$

therefore

$$\alpha_0=-\alpha'(1-\mathrm{L}),\qquad\beta_0=-\beta'(1-\mathrm{L}),\qquad\gamma_0=0,$$

and the principal parts of the components of the magnetic force at the point P due to the oscillator and the assumed surface distribution are given by

$$\alpha=\mathrm{L}\alpha',\qquad\beta=\mathrm{L}\beta',\qquad\gamma=0,$$

where

$$\mathrm{L}=2^{-1/2}e^{1/4\pi\iota}\int_{u_0}^{\infty}e^{-1/2\pi\iota u^2}\,du,$$

and

$$u_0 = \left[\frac{2}{\lambda}\left(\frac{1}{\mathrm{OM}}+\frac{1}{\mathrm{PM}}\right)\right]^{1/2}\mathrm{RM},$$

where λ is the wave-length of the oscillations and M is the foot of the perpendicular from R on OP.

When the point P is outside the tangent cone from O to the surface and near to the tangent cone and at a distance from O greater than OR it appears that the focal lines as determined above are situated one of them close to the surface of the obstacle and the other near to the source, hence the same analysis applies as in the immediately preceding case, and the components of the magnetic force at P due to the assumed surface distribution are given by

$$\alpha_0 = -\mathrm{L}\alpha', \qquad \beta_0 = -\mathrm{L}\beta', \qquad \gamma_0 = 0,$$

where

$$\mathrm{L} = \int^{-\xi_1}\int e^{-\iota\kappa(r_1+r)}\,d\sigma \Big/ \iint e^{-\iota\kappa(r_1+r)}\,d\sigma,$$

and in the integral in the denominator the limits are indefinitely extended, while in the integral in the numerator the upper limit is $-\xi_1$ where $\xi_1$ denotes the perpendicular from R on OP. Evaluating the above, the value of L is given by

$$\mathrm{L} = 2^{-1/2}\,e^{1/4\pi\iota}\int_{-\infty}^{-u_0} e^{-1/2\pi\iota u^2}\,du = 2^{-1/2}\,e^{1/4\pi\iota}\int_{u_0}^{\infty} e^{-1/2\pi\iota u^2}\,du,$$

where

$$u_0 = \left[\frac{2}{\lambda}\left(\frac{1}{\mathrm{OM}}+\frac{1}{\mathrm{PM}}\right)\right]^{1/2}\mathrm{RM},$$

and therefore the principal parts of the components of the magnetic force at the point P outside the tangent cone and near to it are given by

$$\alpha = (1-\mathrm{L})\,\alpha', \qquad \beta = (1-\mathrm{L})\,\beta', \qquad \gamma = 0,$$

where L has the value given above.

Considering now a point P on the surface of the obstacle near to the curve of contact of the tangent cone and on the side of the surface remote from O it appears that the principal parts of the components of the magnetic force due to the oscillator and the assumed surface distribution are $\mathrm{L}\alpha'$, $\mathrm{L}\beta'$, 0, and there will be a corresponding tangential electric force involving the factor L, therefore the surface condition is not satisfied at such points; the same result holds for points near the curve of contact on the side of the surface next O. It is therefore now necessary to determine what the effect of the electric current distribution on the surface which would reduce this unbalanced electric force to zero is. Taking first the case of a point P inside the tangent cone on the side of the surface of the obstacle remote from O and near to the boundary of the cone, with the same notation as above, the components of the magnetic force at P due to the electric current distribution necessary to balance the unbalanced tangential electric force in the neighbourhood of the curve of contact of the tangent cone are, compared with the components of the magnetic force at P due to the oscillator, of the order

$$\mathrm{K} = \frac{\iota\kappa}{2\pi\mathrm{R}}\iint e^{-\iota\kappa(\xi^2+\eta^2)/2\mathrm{R}}\,\mathrm{L}\,d\sigma,$$

where L varies from point to point of the surface.*

The integration with respect to $\eta$ can be effected and it follows that

$$\mathrm{K} = \frac{\iota\kappa}{2\pi\mathrm{R}}\left(\frac{\iota\kappa}{2\mathrm{R}}\right)^{-1/2}\pi^{1/2}\int^{\xi_1} e^{-\iota\kappa\xi^2/2\mathrm{R}}\,\mathrm{L}\,d\xi.$$

Let $\xi_1$ denote the perpendicular from R on OP, $\xi$ the perpendicular from a point $\mathrm{P}_1$ on the surface on OP, then

$$\mathrm{L} = 2^{-1/2}e^{1/4\pi\iota}\int_{u_0}^{\infty} e^{-1/2\pi\iota u^2}\,du,$$

where

$$u_0 = \left(\frac{2}{\lambda\mathrm{P}_1\mathrm{M}_1}\right)^{1/2}\mathrm{RM}_1 = \left[\frac{2(\xi_1-\xi)^{3/2}}{\lambda(2\rho)^{1/2}}\right]^{1/2},$$

in which $\rho$ denotes the radius of curvature of the normal section of the surface through OR, hence

$$\mathrm{L} = 2^{-3/4}e^{1/4\pi\iota}\pi^{-1/2}\kappa^{1/2}\rho^{-1/4}(\xi_1-\xi)^{3/4}\int_1^{\infty} e^{-1/2\iota\kappa(\xi_1-\xi)^{3/2}v^2(2\rho)^{-1/2}}\,dv.$$

Therefore K is at most of the order

$$\left(\frac{\iota\kappa}{2\mathrm{R}}\right)^{1/2}\kappa^{1/2}\rho^{-1/4}\xi_1^{3/4}\int_1^{\infty}\left(\frac{\iota\kappa}{2\mathrm{R}}+\frac{3\iota\kappa v^2}{16(2\rho\xi_1)^{1/2}}\right)^{-1/2} e^{-1/2\iota\kappa\xi_1^{3/2}v^2(2\rho)^{-1/2}}\,dv,$$

if

$$\left[\frac{\iota\kappa}{2\mathrm{R}}+\frac{3\iota\kappa v^2}{16(2\rho\xi_1)^{1/2}}\right]^{1/2}\xi_1$$

is of the same or higher order than unity, that is, provided R is of the same or higher order than $(\rho\xi_1)^{1/2}$, if $\kappa^{1/2}\xi_1$ is of the same or higher order than $(\rho\xi_1)^{1/4}$, that is, if $\xi_1$ is of the same or higher order than $\rho(\kappa\rho)^{-2/3}$. Hence, when R is of the same or higher order than $(\rho\xi_1)^{1/2}$, and $\xi_1$ is of the same or higher order than $\rho(\kappa\rho)^{-2/3}$, the quantity K is at most of the order

$$\kappa^{1/2}\xi_1^{3/4}\rho^{-1/4}\int_1^{\infty} e^{-1/2\iota\kappa\xi_1^{3/2}(2\rho)^{-1/2}v^2}\,dv,$$

which is at most of the order $\kappa^{-1/2}\xi_1^{-3/4}\rho^{1/4}$, for $\kappa\xi_1^{3/2}$ is of the same or higher order than

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* To simplify the analysis $\mathrm{R}_1$ is taken infinite as the result is not affected provided the oscillator is at a distance from the surface of the obstacle comparable with a wave-length.

$\rho^{1/2}$; therefore, when R is of the same or higher order than $(\rho\xi_1)^{1/2}$, and $\xi_1$ is of higher order than $\rho(\kappa\rho)^{-2/3}$, K is of lower order than unity. When R is of higher order than $(\rho\xi_1)^{1/2}$, and $\xi_1$ is of the same or higher order than $\rho(\kappa\rho)^{-2/3}$, the quantity K is at most of the order

$$\kappa^{1/2}\xi_1 \mathrm{R}^{-1/2}\int_1^\infty e^{-1/2\,\iota\kappa\xi_1{}^{3/2}(2\rho)^{-1/2}v^2}\,dv,$$

which is at most of the order $(\rho\xi_1)^{1/4}\mathrm{R}^{-1/2}$, and therefore, when R is of higher order than $(\rho\xi_1)^{1/2}$, and $\xi_1$ is of the same or higher order than $\rho(\kappa\rho)^{-2/3}$, K is of lower order than unity. Hence, at a point P inside the tangent cone from O to the surface of the obstacle, the effect of the electric current distribution which balances the unbalanced tangential electric force due to the oscillator and the assumed surface distribution, and is on the part of the surface inside the tangent cone and remote from O, is of lower order than that due to the oscillator, provided that the perpendicular distance $\xi_1$ of the point of contact R of the tangent from O to the surface in the plane of incidence is of the same or higher order than $\rho(\kappa\rho)^{-2/3}$, and that the distance of the point P from the surface of the obstacle measured along OP is of higher order than $\rho(\kappa\rho)^{-1/3}$, or provided that $\xi_1$ is of higher order than $\rho(\kappa\rho)^{-2/3}$, and the distance of P from the surface is of the same order as $\rho(\kappa\rho)^{-1/3}$. In a similar way it may be shown that the effect of the electric current distribution on the part of the surface of the obstacle next the oscillator which balances the unbalanced tangential electric force there, due to the oscillator and the assumed surface distribution, is of lower order than that due to the oscillator at a point P satisfying similar conditions to those in the previous case. Therefore, at a point P inside the tangent cone from O which satisfies the above conditions the principal parts of the components of the magnetic and electric forces are equal to the principal parts of the components of the magnetic and electric forces due to the oscillator and the assumed electric current surface distribution. The same result holds for points P outside the tangent cone similarly restricted; the region omitted in the neighbourhood of the tangent cone is that lying between the cone through O which cuts the surface of the obstacle in a curve inside the tangent cone on the side of the surface next O and at a distance from the curve of contact of the tangent cone of the order $\rho(\kappa\rho)^{-1/3}$ and a surface touching the obstacle along this surface and generated by straight lines in the plane of incidence at each point of the curve.

The principal parts of the magnetic and electric forces at points on the surface of the obstacle which are at a distance from the curve of contact of the tangent cone of order greater than $\rho(\kappa\rho)^{-1/3}$ can now be found. Let P be a point on the surface inside the tangent cone and on the side of the surface remote from O, and let OP cut the surface between O and P in the point Q, then PQ is of higher order than $\rho(\kappa\rho)^{-1/3}$ if P is at a distance along the surface from the curve of contact of the tangent cone which is of higher order than $\rho(\kappa\rho)^{-1/3}$, and therefore the effects at P of the surface distribution on the part of the surface inside the tangent cone and nearest to O which balances the unbalanced tangential electric force there, due to the oscillator and the assumed surface distribution, is of lower order than that due to the oscillator. Further, if M is the actual magnetic force at P tangential to the surface, the magnetic force at P due to the distribution in the neighbourhood of P is $\frac{1}{2}$M tangential to the surface; now it has been shown that the magnetic force tangential to the surface at P, due to the oscillator and the assumed surface distribution, is LM′, where M′ is the tangential magnetic force due to the oscillator alone and L has the value previously determined, therefore

$$\mathrm{M} = \tfrac{1}{2}\mathrm{M}+\mathrm{LM}', \quad \text{that is,} \quad \mathrm{M} = 2\mathrm{LM}'.$$

Hence, at points P inside the tangent cone on the side of the surface remote from O, the principal part of the magnetic force tangential to the surface is 2LM′, and the principal part of the electric force perpendicular to the surface is 2LE′, where M′, E′ are the principal parts of the magnetic force tangential to the surface and of the electric force perpendicular to the surface due to the oscillator. Similarly at points on the surface on the side nearest to the oscillator the principal part of the tangential magnetic force is $2(1-\mathrm{L})\,\mathrm{M}'$, and the principal part of the electric force perpendicular to the surface is $2(1-\mathrm{L})\,\mathrm{E}'$.

The preceding analysis can be adapted to the case of a conducting screen when the radii of curvature of the screen are large compared with the wave-length of the oscillations. The same assumed distribution as above on the two surfaces of the screen, viz., double the electric current distribution that would produce the magnetic force tangential to the screen on the side on which the waves are incident and a zero electric current distribution on the other side, give the important part of the asymptotic solution of the problem. The analysis only differs from the preceding owing to the presence of edges, and it will be proved that the effect of the actual distribution at the edge differs from that due to the oscillator and the assumed distribution by a quantity of lower order than the corresponding component in the waves due to the oscillator. At an edge the radius of curvature is zero, and therefore the distribution in the neighbourhood of the edge is the same as if the wave-length were indefinitely great; hence the electric current distribution in the immediate neighbourhood if the edge varies as $r_1^{-1/2}$, where $r_1$ is the distance along the screen perpendicular to the edge, that is, the electric current distribution in the neighbourhood of the edge is $(\kappa r_1)^{-1/2}$ multiplied by a quantity of the order of the magnetic force in the incident waves. The effect of this distribution in the neighbourhood of the edge is of the order

$$\mathrm{K}' = \kappa \iint \frac{e^{-\iota\kappa r}}{r\,(\kappa r_1)^{1/2}}\, d\mathrm{S}$$

at a point P compared with the corresponding component in the waves due to the oscillator. Let Q be the point on the edge such that QP is at right angles to the edge, and let QP make an angle $\theta$ with the line QR through Q in the screen perpendicular to the edge, then

$$\mathrm{K}' = \kappa \int \left(\frac{\iota\kappa}{2r}\right)^{-1/2} \frac{e^{-\iota\kappa r}}{r\,(\kappa r_1)^{1/2}}\,dr_1,$$

where $r$ now denotes the distance of P from a point in the line QR; for points near to Q

$$r^2 = \mathrm{R}^2 + r_1^2 - 2r_1\mathrm{R}\cos\theta,$$

where R is the distance QP, and therefore the principal part of K′ is

$$\left(\frac{\iota}{2}\right)^{-1/2} \frac{e^{-\iota\kappa\mathrm{R}}}{\mathrm{R}^{1/2}} \int_0^\infty \frac{e^{\iota\kappa r_1\cos\vartheta}}{r_1^{\,1/2}}\,dr_1,$$

that is, K′ is a quantity of the order $(\kappa\mathrm{R})^{-1/2}$; hence at points which are at a distance from the nearest edge great compared to a wave-length the effect of the edges is negligible in comparison with the waves due to the oscillator. Therefore the principal parts of the components of the electric and magnetic forces at such points are equal to the principal parts of the electric and magnetic forces due to the assumed electric current distribution and the oscillator.

It may be verified that this agrees with the known solution of the problem of the semi-infinite conducting plane. Taking the case where the electric force in the incident waves is parallel to the edge, let the origin be in the edge, the axis of $z$ along the edge, the axis of $y$ perpendicular to the plane of the edge, and let the direction of the incident waves make an angle $\vartheta_1$ with the conducting plane, then the components of the electric force in the incident waves are

$$\mathrm{X} = \mathrm{Y} = 0, \qquad \mathrm{Z} = e^{\iota\kappa(\mathrm{V}t + x\cos\vartheta_1 + y\sin\vartheta_1)}$$

and the components of the assumed electric current distribution on the upper face of the plane are

$$\bar{\alpha} = 0, \qquad \bar{\beta} = 0, \qquad \bar{\gamma} = 2\mathrm{V}^{-1}e^{\iota\kappa(\mathrm{V}t + x\cos\vartheta_1)},$$

and on the lower face

$$\bar{\alpha} = 0, \qquad \bar{\beta} = 0, \qquad \bar{\gamma} = 0.$$

Hence the components of the electric force due to the assumed distribution are

$$\mathrm{X} = 0, \qquad \mathrm{Y} = 0, \qquad \mathrm{Z} = -(2\pi)^{-1}\iota\kappa\sin\vartheta_1 \int_0^\infty \int_{-\infty}^\infty e^{\iota\kappa(\mathrm{V}t + x_1\cos\vartheta_1 - r)}\,r^{-1}\,d\mathrm{S},$$

where $(x_1, 0, z_1)$ is any point on the screen, $r$ is the distance of the point P $(x, y, 0)$ from the point on the screen, and the integration is taken over the conducting plane. Now

$$r^2 = (x - x_1)^2 + y^2 + z_1^2 = \rho^2 + z_1^2,$$

where

$$\rho^2 = (x-x_1)^2+y^2,$$

therefore

$$Z = -(2\pi)^{-1}\iota\kappa \sin \vartheta_1 \int_0^\infty \int_{-\infty}^\infty e^{\iota\kappa(Vt+x_1 \cos \vartheta_1-\rho-{}^1\!/_2 z_1{}^2/\rho-)} r^{-1}\, dx_1\, dz_1,$$

whence the principal part of Z is given by the principal part of $Z_0$, where

$$Z_0 = -(2\pi)^{-1}\iota\kappa \sin \vartheta_1 \int_0^\infty (\iota\kappa/2\rho)^{-1/2} \rho^{-1} e^{\iota\kappa(Vt+x_1 \cos \vartheta_1-\rho)}\, dx_1;$$

now $x_1 \cos \vartheta_1-\rho$ is stationary when

$$x = x_1-\rho \cos \vartheta_1, \qquad y = \pm\rho \sin \vartheta_1,$$

the upper sign corresponds to the case where P is on the positive side of the screen and the lower sign to the case where P is on the negative side of the screen. When P is on the negative side of the screen, let the line through P parallel to the direction of the waves cut the screen or the plane of the screen produced at the point Q $(\xi_1, 0, 0)$, then, writing $x_1 = \xi+\xi_1$ and R for the distance QP,

$$\rho^2 = R^2+2\xi R \cos \vartheta_1+\xi^2,$$
$$x_1 \cos \vartheta_1-\rho = \xi_1 \cos \vartheta_1-R-\tfrac{1}{2}\xi^2 \sin^2 \vartheta_1/R,$$

and the principal part of $Z_0$ is given by

$$-(2\pi)^{-1}(2\iota\kappa)^{1/2}R^{-1/2} \sin \vartheta_1\, e^{\iota\kappa(Vt-R+\xi_1 \cos \vartheta_1)} \int_{-\xi_1}^\infty e^{-1/2\,\iota\kappa\xi^2 \sin^2 \vartheta_1/R}\, d\xi,$$

that is, by

$$-2^{-1/2} e^{1/4\,\pi\iota} e^{\iota\kappa(Vt+x \cos \vartheta_1+y \sin \vartheta_1)} \int_{-u_0}^\infty e^{-1/2\,\pi\iota u^2}\, du,$$

where

$$u_0 = 2^{1/2}(\lambda R)^{-1/2}\xi_1 \sin \vartheta_1,$$

and therefore the electric force at a point on the negative side of the screen is given by

$$Z = 2^{-1/2} e^{1/4\,\pi\iota} e^{\iota\kappa(Vt+x \cos \vartheta_1+y \sin \vartheta_1)} \int_{-\infty}^{-u_0} e^{-1/2\,\pi\iota u^2}\, du.$$

When the point P is on the positive side of the screen, the principal part of the electric force due to the assumed distribution is found in the same way to be

$$-2^{-1/2} e^{1/4\,\pi\iota} e^{\iota\kappa(Vt+x \cos \vartheta_1-y \sin \vartheta_1)} \int_{-u_0}^\infty e^{-1/2\,\pi\iota u^2}\, du,$$

where the point Q $(\xi_1, 0)$ is the point where the line through P in the plane perpendicular to the edge making an angle $\pi-\vartheta_1$, with the plane of the screen meets it and therefore the electric force at a point P on the positive side of the screen is given by

$$Z = e^{\iota\kappa(Vt+x \cos \vartheta_1+y \sin \vartheta_1)}-2^{-1/2} e^{1/4\,\pi\iota} e^{\iota\kappa(Vt+x \cos \vartheta_1-y \sin \vartheta_1)} \int_{-u_0}^\infty e^{-1/2\,\pi\iota u^2}\, du.$$

When the point P is at a distance from the edge of the screen great compared with a wave-length these expressions are sensibly the same as those of the known accurate solution, as $u_0$ is very large unless the line OP makes an angle which is very small with the line through O making an angle $\pi+\vartheta_1$ with the plane of the screen, or with the line through O making an angle $\pi-\vartheta_1$ with the plane of the screen, and in these cases the results are the same.

When the magnetic force in the incident waves is parallel to the edge of the conducting screen

$$\alpha = 0, \qquad \beta = 0, \qquad \gamma = e^{\iota\kappa(\mathrm{V}t+x\cos\vartheta_1+y\sin\vartheta_1)}$$

in the incident waves, and the assumed electric current distribution on the positive side of the screen is given by

$$\bar{\alpha} = 2e^{\iota\kappa(\mathrm{V}t+x\cos\vartheta_1)}, \qquad \bar{\beta} = 0, \qquad \bar{\gamma} = 0,$$

and on the negative side of the screen by

$$\bar{\alpha} = 0, \qquad \bar{\beta} = 0, \qquad \bar{\gamma} = 0,$$

whence the components of the magnetic force due to the assumed surface distribution at a point P $(x, y, z)$ are

$$\alpha = 0, \qquad \beta = 0, \qquad \gamma = -\frac{1}{2\pi}\int_0^\infty\int_{-\infty}^\infty \frac{\partial}{\partial y}\frac{e^{\iota\kappa(\mathrm{V}t+x_1\cos\vartheta_1-r)}}{r}\,dx_1, dz_1,$$

and, by a similar analysis to that in the preceding case, when the point P is on the negative side of the screen

$$\gamma = -e^{\iota\kappa(\mathrm{V}t+x\cos\vartheta_1+y\sin\vartheta_1)}\,2^{-1/2}\,e^{1/4\pi\iota}\int_{-u_0}^\infty e^{-1/2\pi\iota u^2}\,du,$$

where

$$u_0 = 2^{1/2}(\lambda\mathrm{R})^{-1/2}\,\xi_1\sin\vartheta_1,$$

$\xi_1$ having the same meaning as above, and therefore the resultant magnetic force on the negative side of the screen is given by

$$e^{\iota\kappa(\mathrm{V}t+x\cos\vartheta_1+y\sin\vartheta_1)}\,2^{-1/2}\,e^{1/4\pi\iota}\int_{-\infty}^{-u_0} e^{-1/2\pi\iota u^2}\,du.$$

When the point P is on the positive side of the screen the magnetic force is given by

$$e^{\iota\kappa(\mathrm{V}t+x\cos\vartheta_1+y\sin\vartheta_1)}+e^{\iota\kappa(\mathrm{V}t+x\cos\vartheta_1-y\sin\vartheta_1)}\,2^{-1/2}\,e^{1/4\pi\iota}\int_{-u_0}^\infty e^{-1/2\pi\iota u^2}\,du,$$

these results again agreeing with the known accurate solution when the point P is at a distance from the edge great compared with a wave-length. Again, when the conducting screen is an infinite plane with a slit in it of breadth $2d$ bounded by parallel edges, taking the origin in the middle line of the slit, this line being the axis of $z$, and the straight line in the plane of the screen perpendicular to it the axis of $x$, the same analysis shows that, when the electric force in the incident waves is parallel to the edges of the slit and is given by

$$Z = e^{\iota\kappa(\mathrm{V}t+x\cos\vartheta_1+y\sin\vartheta_1)}$$

the electric force at a point P on the negative side of the screen is given by

$$Z = e^{\iota\kappa(\mathrm{V}t+x\cos\vartheta_1+y\sin\vartheta_1)}\,2^{-1/2}\,e^{1/4\,\pi\iota}\int_{-u_0}^{-u_1} e^{-1/2\,\pi\iota u^2}\,du,$$

where

$$u_0 = 2^{1/2}(\lambda\mathrm{R})^{-1/2}(\xi_0+d)\sin\vartheta_1,$$
$$u_1 = 2^{1/2}(\lambda\mathrm{R})^{-1/2}(\xi_1-d)\sin\vartheta_1,$$

and $(\xi_1, 0)$ are the co-ordinates of the point Q where the straight line through P parallel to the direction of the incident waves cuts the plane of the screen. The electric force at a point P on the positive side of the screen is given by

$$Z = e^{\iota\kappa(\mathrm{V}t+x\cos\vartheta_1+y\sin\vartheta_1)} - e^{\iota\kappa(\mathrm{V}t+x\cos\vartheta_1-y\sin\vartheta_1)} + e^{\iota\kappa(\mathrm{V}t+x\cos\vartheta_1-y\sin\vartheta_1)}\,2^{-1/2}\,e^{1/4\,\pi\iota}\int_{-u_0}^{-u_1} e^{-1/2\,\pi\iota u^2}\,du,$$

where $u_0$, $u_1$ are the same as above and $\xi_1$, 0 are the co-ordinates of the point where the straight line through P in a plane perpendicular to the edges making an angle $\pi-\vartheta_1$ with the plane of the screen cuts the screen.

When the magnetic force in the incident waves is parallel to the edges and is given by

$$\gamma = e^{\iota\kappa(\mathrm{V}t+x\cos\vartheta_1+y\sin\vartheta_1)},$$

the magnetic force at a point P on the negative side of the screen is given by

$$\gamma = e^{\iota\kappa(\mathrm{V}t+x\cos\vartheta_1+y\sin\vartheta_1)}\,2^{-1/2}\,e^{1/4\pi\iota}\int_{-u_0}^{-u_1} e^{-1/2\pi\iota u^2}\,du,$$

and at a point P on the positive side of the screen by

$$\gamma = e^{\iota\kappa(\mathrm{V}t+x\cos\vartheta_1+y\sin\vartheta_1)} + e^{\iota\kappa(\mathrm{V}t+x\cos\vartheta_1-y\sin\vartheta_1)} - e^{\iota\kappa(\mathrm{V}t+x\cos\vartheta_1-y\sin\vartheta_1)}\,2^{-1/2}\,e^{1/4\,\pi\iota}\int_{-u_0}^{-u_1} e^{-1/2\,\pi\iota u^2}\,du,$$

where $u_0$, $u_1$ in each case have the same meanings as in the corresponding case in the preceding. The discussion of these results is similar to that given by GILBERT* for an absorbing screen.

The corresponding results for a plane conducting screen in which there is any number of parallel slits can be written down immediately, provided the breadth of the portion of the screen between every two consecutive slits is great compared with a wave-length; when the number of conducting strips in the breadth of a wave-length is large the effect of the edges becomes the most important.

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* 'Memoires couronnés de l'Acad. de Bruxelles,' t. xxxi., p. 29, 1863.

As a further example, the case of an infinite plane conducting screen with a circular aperture of radius $a$ will be worked out. As above, let a straight line through the centre of the aperture perpendicular to the plane of the screen be the axis of $y$, the axes of $x$ and $z$ being in the plane of the screen, and let the incident waves be given by

$$\alpha = 0, \qquad \beta = 0, \qquad \gamma = e^{\iota\kappa(\mathrm{V}t+y)},$$

the magnetic force being parallel to the plane of the screen and the direction of the waves perpendicular to it; then the components of the magnetic force due to the distribution to be assumed over the screen are

$$\alpha_0 = 0, \qquad \beta_0 = \frac{1}{2\pi} e^{\iota\kappa\mathrm{V}t} \iint \frac{\partial}{\partial z} \frac{e^{-\iota\kappa r}}{r} \, d\mathrm{S}, \qquad \gamma_0 = -\frac{1}{2\pi} e^{\iota\kappa\mathrm{V}t} \iint \frac{\partial}{\partial y} \frac{e^{-\iota\kappa r}}{r} \, d\mathrm{S},$$

where $r$ is the distance of the point from a point in the plane of the screen and the integrals are taken over the conducting part of it. Transforming to cylindrical co-ordinates

$$r^2 = y^2 + \rho^2 + \rho_1^{\,2} - 2\rho\rho_1 \cos(\phi_1 - \phi),$$

where the co-ordinates of the point P are $(\rho, \phi, y)$ and of a point in the plane $(\rho_1, \phi_1, 0)$. The principal parts of the integrals are contributed by the elements in the neighbourhood of the point for which $r$ is stationary, that is, when

$$\phi_1 = \phi, \qquad \rho_1 = \rho,$$

and then, provided $\rho$ is not small,

$$\iint \frac{e^{-\iota\kappa r}}{r} \, d\mathrm{S} = \int_a^\infty \int_0^{2\pi} \frac{e^{-\iota\kappa r}}{r} \rho_1 \, d\rho_1 \, d\phi_1 = \pi^{1/2} \left(\frac{\iota\kappa\rho}{2}\right)^{-1/2} \int_a^\infty e^{-\iota\kappa r_1} r_1^{\,-1/2} \rho_1^{\,1/2} \, d\rho_1,$$

where $r_1^{\,2} = y^2 + (\rho_1 + \rho)^2$.

For a point on the negative side of the screen $y = -d$, and writing $\rho_1 = \rho + \rho'$, the principal part of the integral $\iint e^{-\iota\kappa r} r^{-1} \, d\mathrm{S}$ is given by

$$(2\pi)^{1/2} (\iota\kappa\rho)^{-1/2} \int_{a-\rho}^{\infty} e^{-\iota\kappa d - 1/2\,\iota\kappa\rho'^2/d} \, d^{-1/2} (\rho + \rho')^{1/2} \, d\rho',$$

that is, provided $\kappa\rho^2$ is great compared with $d$, by

$$2\pi (\iota\kappa)^{-1} 2^{-1/2} e^{1/4\,\pi\iota} e^{\iota\kappa y} \int_{u_0}^{\infty} e^{-1/2\,\pi\iota u^2} \, du,$$

where $u_0 = 2^{1/2} (\lambda d)^{-1/2} (a - \rho)$.

Hence, at a point P on the negative side of screen, the principal parts of the components of the magnetic force due to the assumed distribution are

$$\alpha_0 = 0,$$
$$\beta_0 = (\iota\kappa)^{-1} 2^{-1/2} e^{1/4\pi\iota} e^{\iota\kappa(\mathrm{V}t+y)} \frac{\partial}{\partial z} \int_{u_0}^{\infty} e^{-1/2\pi\iota u^2}\, du,$$
$$\gamma_0 = -2^{-1/2} e^{1/4\pi\iota} e^{\iota\kappa(\mathrm{V}t+y)} \int_{u_0}^{\infty} e^{-1/2\pi\iota u^2}\, du,$$

and therefore the principal parts of the components of the magnetic force at the point P are

$$\alpha = 0, \qquad \beta = 0, \qquad \gamma = 2^{-1/2} e^{1/4\pi\iota} e^{\iota\kappa(\mathrm{V}t+y)} \int_{-\infty}^{u_0} e^{-1/4\pi\iota u^2}\, du,$$

for $\beta_0$ is of order $(\kappa d)^{-1/2}$ compared with the magnetic force in the incident waves. When $\rho$ is small compared with $a$,

$$\int_a^{\infty}\int_0^{2\pi} e^{-\iota\kappa r} r^{-1} \rho_1\, d\rho_1\, d\phi_1 = d^{-1} \int_a^{\infty}\int_0^{2\pi} e^{-\iota\kappa d + \iota\kappa\rho\rho_1 d^{-1}\cos(\phi_1-\phi) - 1/2\iota\kappa\rho_1{}^2/d} \rho_1\, d\rho_1\, d\phi_1,$$

that is, the principal part only being retained,

$$\iint e^{-\iota\kappa r} r^{-1}\, d\mathrm{S} = 2\pi d^{-1} e^{\iota\kappa y} \int_a^{\infty} e^{-1/2\iota\kappa\rho_1{}^2/d} \mathrm{J}_0(\kappa\rho\rho_1/d)\, \rho_1\, d\rho_1,$$

or

$$\iint e^{-\iota\kappa r} r^{-1}\, d\mathrm{S} = 2\pi(\iota\kappa)^{-1} e^{\iota\kappa y} - 2\pi d^{-1} e^{\iota\kappa y} \int_0^{a} e^{-1/2\iota\kappa\rho_1{}^2/d} \mathrm{J}_0(\kappa\rho\rho_1/d)\, \rho_1\, d\rho_1.$$

Therefore, when $\rho$ is small, the principal parts of the components of the magnetic force at a point near the axis of $y$ on the negative side of the screen are given by

$$\alpha = 0,$$
$$\beta = \kappa z \rho^{-1} d^{-2} e^{\iota\kappa(\mathrm{V}t+y)} \int_0^{a} e^{-1/2\iota\kappa\rho_1{}^2/d} \mathrm{J}_1(\kappa\rho\rho_1/d)\, \rho_1{}^2\, d\rho_1,$$
$$\gamma = \iota\kappa d^{-1} e^{\iota\kappa(\mathrm{V}t+y)} \int_0^{a} e^{-1/2\iota\kappa\rho_1{}^2/d} \mathrm{J}_0(\kappa\rho\rho_1/d)\, \rho_1\, d\rho_1.$$

As an example illustrating the phenomenon of the bright spot, the case of a conducting circular disc of radius $a$ with a Hertzian oscillator on the axis will be treated. Let the axis of the disc be the axis of $z$, the origin being at the oscillator and the equation of the plane of the disc $z = z_0$; the components of the magnetic force due to the oscillator are

$$\alpha' = \frac{\kappa}{\mathrm{V}} \frac{\partial}{\partial\eta} \frac{e^{\iota\kappa(\mathrm{V}t-r_1)}}{r_1}, \qquad \beta' = -\frac{\kappa}{\mathrm{V}} \frac{\partial}{\partial\xi} \frac{e^{\iota\kappa(\mathrm{V}t-r_1)}}{r_1}, \qquad \gamma' = 0,$$

where $\xi$, $\eta$, $\zeta$ are the co-ordinates of any point and

$$r_1{}^2 = \xi^2 + \eta^2 + \zeta^2,$$

and the components of the electric current distribution to be assumed on the lower side of the disc are

$$\bar{\alpha} = -\frac{2\kappa}{\mathrm{V}}\frac{\partial}{\partial\xi}\frac{e^{\iota\kappa(\mathrm{V}t-r_1)}}{r_1}, \qquad \bar{\beta} = -\frac{2\kappa}{\mathrm{V}}\frac{\partial}{\partial\eta}\frac{e^{\iota\kappa(\mathrm{V}t-r_1)}}{r_1}, \qquad \bar{\gamma} = 0,$$

where $\xi$, $\eta$, $z_0$ are now the co-ordinates of a point on the disc. Therefore the components of the magnetic force at a point $x$, $y$, $z$ due to this distribution are

$$\alpha = \frac{\kappa}{2\pi\mathrm{V}}\iint\frac{\partial}{\partial z}\frac{1}{r}\frac{\partial}{\partial\xi}\frac{1}{r_1}e^{\iota\kappa(\mathrm{V}t-r_1-r)}\,d\mathrm{S},$$

$$\beta = -\frac{\kappa}{2\pi\mathrm{V}}\iint\frac{\partial}{\partial z}\frac{1}{r}\frac{\partial}{\partial\eta}\frac{1}{r_1}e^{\iota\kappa(\mathrm{V}t-r_1-r)}\,d\mathrm{S},$$

$$\gamma = 0,$$

where

$$r^2 = (x-\xi)^2+(y-\eta)^2+(z-z_0)^2,$$

writing

$$x = \rho\cos\phi, \qquad y = \rho\sin\phi, \qquad \xi = \rho_1\cos\phi_1, \qquad \eta = \rho_1\sin\phi_1,$$

the resultant magnetic force due to the distribution is

$$\beta\cos\phi-\alpha\sin\phi = \frac{\iota\kappa^2}{2\pi\mathrm{V}}\frac{\partial}{\partial z}\int_0^a\int_0^{2\pi}\sin(\phi_1+\phi)\frac{\rho_1^2}{r_1^2 r}e^{\iota\kappa(\mathrm{V}t-r_1-r)}\,d\rho_1\,d\phi_1,$$

where

$$r^2 = (z-z_0)^2+\rho^2+\rho_1^2-2\rho\rho_1\cos(\phi_1-\phi), \qquad r_1^2 = z_0^2+\rho_1^2.$$

Now $r_1+r$ is stationary when

$$\phi_1 = \phi, \qquad (\rho-\rho_1)/r = \rho_1/r_1,$$

and, writing R, $\mathrm{R}_1$ for the corresponding values of $r$, $r_1$ and $\sin\vartheta = \rho_1/r_1$, the principal value of the magnetic force is given by

$$\frac{\iota\kappa^2}{2\pi\mathrm{V}}\left(\frac{\iota\kappa}{2}\right)^{-1/2}\pi^{1/2}\sin 2\phi\,\frac{\partial}{\partial z}e^{\iota\kappa(\mathrm{V}t-\mathrm{R}_1-\mathrm{R})}\int_{-\rho_1}^{a-\rho_1}\rho^{-1/2}\rho_1^{3/2}\mathrm{R}_1^{-2}\mathrm{R}^{-1/2}e^{-1/2\,\iota\kappa\cos^2\vartheta\,(\mathrm{R}_1^{-1}+\mathrm{R}^{-1})\rho'^2}\,d\rho'.$$

When $a-\rho_1$ is not small, the principal value of the above expression is

$$\frac{\kappa}{\mathrm{V}}\frac{\partial}{\partial z}\frac{e^{\iota\kappa(\mathrm{V}t-\mathrm{R}_1-\mathrm{R})}}{\mathrm{R}_1+\mathrm{R}}\sin 2\phi\tan\vartheta,$$

that is, when the point $x$, $y$, $z$ is on the upper side of the disc

$$-\frac{\iota\kappa^2}{\mathrm{V}}\sin\vartheta\sin 2\phi\,\frac{e^{\iota\kappa(\mathrm{V}t-\mathrm{R}_1-\mathrm{R})}}{\mathrm{R}_1+\mathrm{R}},$$

which is the value, with the opposite sign, of the resultant magnetic force due to the oscillator, and therefore at these points the principal part of the magnetic force vanishes. When $a-\rho_1$ is small, the principal value of the magnetic force due to the assumed distribution is

$$\frac{\kappa}{\mathrm{V}} 2^{-1/2} e^{1/4\pi\iota} \frac{\partial}{\partial z} \frac{e^{\iota\kappa(\mathrm{V}t-\mathrm{R}_1-\mathrm{R})}}{\mathrm{R}_1+\mathrm{R}} \sin 2\phi \tan \vartheta \int_{-\infty}^{u_0} e^{-1/2\pi\iota u^2} du,$$

where

$$u_0 = \left[\frac{2}{\lambda}\left(\frac{1}{\mathrm{R}_1}+\frac{1}{\mathrm{R}}\right)\right]^{1/2} (a-\rho_1) \cos \vartheta,$$

$\rho_1$ being the distance from the axis of the point where the line OP cuts the plane of the disc; whence, if the point P $(x, y, z)$ is on the upper side of the disc, the above has the value

$$-\frac{\iota\kappa^2}{\mathrm{V}} \sin \vartheta \sin 2\phi \frac{e^{\iota\kappa(\mathrm{V}t-\mathrm{R}_1-\mathrm{R})}}{\mathrm{R}_1+\mathrm{R}} 2^{-1/2} e^{1/4\pi\iota} \int_{-\infty}^{u_0} e^{-1/2\pi\iota u^2} du,$$

and therefore the resultant magnetic force at the point P is

$$\frac{\iota\kappa^2}{\mathrm{V}} \sin \vartheta \sin 2\phi \frac{e^{\iota\kappa(\mathrm{V}t-\mathrm{R}_1-\mathrm{R})}}{\mathrm{R}_1+\mathrm{R}} 2^{-1/2} e^{1/4\pi\iota} \int_{u_0}^{\infty} e^{-1/2\pi\iota u^2} du.$$

In the immediately preceding investigation it has been assumed that $\rho$ is not small; when $\rho$ is small and $z > z_0$,

$$r = z-z_0+\tfrac{1}{2}\rho_1^2/(z-z_0)-\{\rho\rho_1 \cos(\phi_1-\phi)\}/(z-z_0)\ldots,$$
$$r_1 = z_0+\tfrac{1}{2}\rho_1^2/z_0,$$

and

$\beta \cos \phi-\alpha \sin \phi$

$$= \frac{\iota\kappa^2}{2\pi\mathrm{V}} \frac{\partial}{\partial z} \int_0^a \int_0^{2\pi} \sin(\phi_1+\phi)\rho_1^2 r_1^{-2} r^{-1} e^{\iota\kappa\{\mathrm{V}t-z+[\rho\rho_1 \cos(\phi_1-\phi)]/(z-z_0)-1/2\rho_1^2 \mid 1/z_0+1/(z-z_0)\}} d\rho_1\, d\phi_1,$$

that is

$\beta \cos \phi-\alpha \sin \phi$

$$= \frac{\iota\kappa^2}{2\pi\mathrm{V}} \frac{\partial}{\partial z} \int_0^a 2\pi\iota \sin 2\phi\, e^{\iota\kappa\{\mathrm{V}t-z-1/2\rho_1^2[1/z_0+1/(z-z_0)]\}} \rho_1^2 (z-z_0)^{-1} z_0^{-2} \mathrm{J}_1\{\kappa\rho\rho_1/(z-z_0)\}\, d\rho_1,$$

or

$$\beta \cos \phi-\alpha \sin \phi = \frac{\iota\kappa^3}{\mathrm{V}} e^{\iota\kappa(\mathrm{V}t-z)} \sin 2\phi\,.\, z_0^{-2}(z-z_0)^{-1} \int_0^a e^{-1/2\iota\kappa\rho_1^2[1/z_0+1/(z-z_0)]} \mathrm{J}_1\{\kappa\rho\rho_1/(z-z_0)\}\rho_1^2\, d\rho_1.$$

This quantity tends to zero with $\rho$, but increases rapidly as $\rho$ increases from zero, thus the resultant magnetic and electric forces at a point on the axis have the same principal values as if there were no disc, and diminish rapidly in the neighbourhood of the axis. Similar results hold for any surface of revolution or for a cylindrical obstacle with a line source parallel to its axis.

### *A Perfectly Absorbing Obstacle.*

When the obstacle is perfectly absorbing, the conditions to be satisfied at its surface are that the electric and magnetic forces in the incident waves are annihilated there. This requires an electric current distribution on the surface which will produce the magnetic force tangential to the surface of the obstacle in the incident waves and a magnetic current distribution on the surface which will produce the electric force tangential to the surface of the obstacle in the incident waves. As in the former case, it is sufficient to discuss the case of a Hertzian oscillator at a point outside the obstacle. Taking the origin of co-ordinates at the oscillator and the axis of $z$ coincident with the axis of the oscillator, the components of the electric and magnetic forces at a point $\xi$, $\eta$, $\zeta$ due to the oscillator are given by

$$\mathrm{X}' = -\iota\frac{\partial^2}{\partial\xi\,\partial\zeta}e^{\iota\kappa(\mathrm{V}t-r_1)}/r_1, \qquad \mathrm{Y}' = -\iota\frac{\partial^2}{\partial\eta\,\partial\zeta}e^{\iota\kappa(\mathrm{V}t-r_1)}/r_1, \qquad \mathrm{Z}' = \iota\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)e^{\iota\kappa(\mathrm{V}t-r_1)}/r_1\,;$$

$$\alpha' = \frac{\kappa}{\mathrm{V}}\frac{\partial}{\partial\eta}e^{\iota\kappa(\mathrm{V}t-r_1)}/r_1, \qquad \beta' = -\frac{\kappa}{\mathrm{V}}\frac{\partial}{\partial\xi}e^{\iota\kappa(\mathrm{V}t-r_1)}/r_1, \qquad \gamma' = 0,$$

where

$$r_1^2 = \xi^2+\eta^2+\zeta^2,$$

and therefore the components of the electric current and magnetic current distributions to be assumed on the portion of the surface of the obstacle on which waves are incident are given by

$$(m\gamma'-n\beta'), \qquad (n\alpha'-l\gamma'), \qquad (l\beta'-m\alpha')\,;$$
$$(m\mathrm{Z}'-n\mathrm{Y}'), \qquad (n\mathrm{X}'-l\mathrm{Z}'), \qquad (l\mathrm{Y}'-m\mathrm{X}')\,;$$

where $l$, $m$, $n$ are the direction cosines of the normal to the surface, these are equivalent to

$$-\iota\kappa^2 n\xi e^{\iota\kappa(\mathrm{V}t-r_1)}/r_1^2\mathrm{V}, \qquad -\iota\kappa^2 n\eta e^{\iota\kappa(\mathrm{V}t-r_1)}/r_1^2\mathrm{V}, \qquad \iota\kappa^2(l\xi+m\eta)\,e^{\iota\kappa(\mathrm{V}t-r_1)}/r_1^2\mathrm{V}\,;$$

$$-\iota\kappa^2\{m(\xi^2+\eta^2)+n\eta\zeta\}\,e^{\iota\kappa(\mathrm{V}t-r_1)}/r_1^2, \qquad \iota\kappa^2\{n\zeta\xi+l(\xi^2+\eta^2)\}\,e^{\iota\kappa(\mathrm{V}t-r_1)}/r_1^2,$$
$$\iota\kappa^2(l\eta\zeta-m\zeta\xi)\,e^{\iota\kappa(\mathrm{V}t-r_1)}/r_1^2,$$

where only the principal parts have been retained; writing

$$\xi_1/r_1 = \lambda_1, \qquad \eta_1/r_1 = \mu_1, \qquad \zeta/r_1 = \nu_1,$$

these become

$$-\iota\kappa^2 n\lambda_1\,e^{\iota\kappa(\mathrm{V}t-r_1)}/r_1\mathrm{V}, \qquad -\iota\kappa^2 n\mu_1\,e^{\iota\kappa(\mathrm{V}t-r_1)}/r_1\mathrm{V}, \qquad \iota\kappa^2(l\lambda_1+m\mu_1)\,e^{\iota\kappa(\mathrm{V}t-r_1)}/r_1,$$

$$-\iota\kappa^2\{m(\lambda_1^2+\mu_1^2)+n\mu_1\nu_1\}\,e^{\iota\kappa(\mathrm{V}t-r_1)}/r_1, \qquad \iota\kappa^2\{n\nu_1\lambda_1+l(\lambda_1^2+\mu_1^2)\}\,e^{\iota\kappa(\mathrm{V}t-r_1)}/r_1,$$
$$\iota\kappa^2(l\mu_1-m\lambda_1)\,\nu_1\,e^{\iota\kappa(\mathrm{V}t-r_1)}/r_1.$$

The components of the electric force at the point $x$, $y$, $z$ due to this distribution are

$$\mathrm{X} = -\frac{\mathrm{V}}{4\pi\iota\kappa}\iint\left[\frac{\partial^2}{\partial x^2}n\lambda_1+\frac{\partial^2}{\partial x\,\partial y}n\mu_1-\frac{\partial^2}{\partial x\,\partial z}(l\lambda_1+m\mu_1)-\frac{1}{\mathrm{V}^2}\frac{\partial^2}{\partial t^2}n\lambda_1\right]\frac{\iota\kappa^2}{rr_1\mathrm{V}}e^{\iota\kappa(\mathrm{V}t-r_1-r)}\,d\mathrm{S}$$

$$-\frac{1}{4\pi}\iint\left[-\frac{\partial}{\partial y}\nu_1(l\mu_1-m\lambda_1)+\frac{\partial}{\partial z}\{n\nu_1\lambda_1+l(\lambda_1^2+\mu_1^2)\}\right]\frac{\iota\kappa^2}{rr_1}e^{\iota\kappa(\mathrm{V}t-r_1-r)}\,d\mathrm{S},$$

$$\mathrm{Y} = -\frac{\mathrm{V}}{4\pi\iota\kappa}\iint\left[\frac{\partial^2}{\partial x\,\partial y}n\lambda_1+\frac{\partial^2}{\partial y^2}n\mu_1-\frac{\partial^2}{\partial y\,\partial z}(l\lambda_1+m\mu_1)-\frac{1}{\mathrm{V}^2}\frac{\partial^2}{\partial t^2}n\mu_1\right]\frac{\iota\kappa^2}{rr_1\mathrm{V}}e^{\iota\kappa(\mathrm{V}t-r_1-r)}\,d\mathrm{S}$$

$$-\frac{1}{4\pi}\iint\left[\frac{\partial}{\partial z}\{m(\lambda_1^2+\mu_1^2)+n\mu_1\nu_1\}+\frac{\partial}{\partial x}\nu_1(l\mu_1-m\lambda_1)\right]\frac{\iota\kappa^2}{rr_1}e^{\iota\kappa(\mathrm{V}t-r_1-r)}\,d\mathrm{S},$$

$$\mathrm{Z} = -\frac{\mathrm{V}}{4\pi\iota\kappa}\iint\left[\frac{\partial^2}{\partial x\,\partial z}n\lambda_1+\frac{\partial^2}{\partial y\,\partial z}n\mu_1-\frac{\partial^2}{\partial z^2}(l\lambda_1+m\mu_1)+\frac{1}{\mathrm{V}^2}\frac{\partial^2}{\partial t^2}(l\lambda_1+m\mu_1)\right]\frac{\iota\kappa^2}{rr_1\mathrm{V}}e^{\iota\kappa(\mathrm{V}t-r_1-r)}\,d\mathrm{S}$$

$$-\frac{1}{4\pi}\iint\left[-\frac{\partial}{\partial x}\{n\nu_1\lambda_1+l(\lambda_1^2+\mu_1^2)\}-\frac{\partial}{\partial y}\{m(\lambda_1^2+\mu_1^2)+n\mu_1\nu_1\}\right]\frac{\iota\kappa^2}{rr_1}e^{\iota\kappa(\mathrm{V}t-r_1-r)}\,d\mathrm{S};$$

retaining only the principal parts and writing

$$(x-\xi)/r = \lambda_2, \qquad (y-\eta)/r = \mu_2, \qquad (z-\zeta)/r = \nu_2,$$

these are equivalent to

$$\mathrm{X} = \frac{\kappa^3}{4\pi}\iint[-\nu_2(\lambda_1+\lambda_2)(l\lambda_1+m\mu_1+n\nu_1)+n\lambda_2(\lambda_1\lambda_2+\mu_1\mu_2+\nu_1\nu_2)-n\lambda_1$$

$$+(l\mu_1-m\lambda_1)(\mu_2\nu_1-\mu_1\nu_2)]\frac{e^{\iota\kappa(\mathrm{V}t-r_1-r)}}{r_1r}\,d\mathrm{S},$$

$$\mathrm{Y} = \frac{\kappa^3}{4\pi}\iint[-\nu_2(\mu_1+\mu_2)(l\lambda_1+m\mu_1+n\nu_1)+n\mu_2(\lambda_1\lambda_2+\mu_1\mu_2+\nu_1\nu_2)-n\mu_1$$

$$+(l\mu_1-m\lambda_1)(\nu_2\lambda_1-\nu_1\lambda_2)]\frac{e^{\iota\kappa(\mathrm{V}t-r_1-r)}}{r_1r}\,d\mathrm{S},$$

$$\mathrm{Z} = \frac{\kappa^3}{4\pi}\iint[\{\lambda_2(\lambda_1+\lambda_2)+\mu_2(\mu_1+\mu_2)\}(l\lambda_1+m\mu_1+n\nu_1)+n\nu_2(\lambda_1\lambda_2+\mu_1\mu_2+\nu_1\nu_2)-n\nu_1$$

$$+(l\mu_1-m\lambda_1)(\lambda_2\mu_1-\mu_2\lambda_1)]\frac{e^{\iota\kappa(\mathrm{V}t-r_1-r)}}{r_1r}\,d\mathrm{S}.$$

The principal parts of these integrals are contributed by the elements in the neighbourhood of the points for which $r_1+r$ is stationary, and, as in the case of the conducting obstacle, there are the two cases when the point Q, for which $r+r_1$ is stationary, is on the straight line OP and when the lines OQ, QP make equal angles with the normal to the surface at Q. In the first case, the point P is on the side of the obstacle remote from the oscillator and at the point Q

$$\lambda_2 = \lambda_1, \qquad \mu_2 = \mu_1, \qquad \nu_2 = \nu_1,$$

and the principal parts of the components of the electric force due to the distribution are

$$X = -\frac{\kappa^3\nu_1\lambda_1}{2\pi}\iint(l\lambda_1+m\mu_1+n\nu_1)\frac{e^{\iota\kappa(Vt-r_1-r)}}{rr_1}dS,$$

$$Y = -\frac{\kappa^3\nu_1\mu_1}{2\pi}\iint(l\lambda_1+m\mu_1+n\nu_1)\frac{e^{\iota\kappa(Vt-r_1-r)}}{rr_1}dS,$$

$$Z = \frac{\kappa^3(\lambda_1^2+\mu_1^2)}{2\pi}\iint(l\lambda_1+m\mu_1+n\nu_1)\frac{e^{\iota\kappa(Vt-r_1-r)}}{rr_1}dS,$$

that is

$$X = \frac{\kappa^3\nu_1\lambda_1}{2\pi R_1 R}\iint e^{\iota\kappa(Vt-r_1-r)}\,d\sigma,$$

$$Y = \frac{\kappa^3\nu_1\mu_1}{2\pi R_1 R}\iint e^{\iota\kappa(Vt-r_1-r)}\,d\sigma,$$

$$Z = -\frac{\kappa^3(\lambda_1^2+\mu_1^2)}{2\pi R_1 R}\iint e^{\iota\kappa(Vt-r_1-r)}\,d\sigma,$$

where $OP = R_1$, $QP = R$, and $d\sigma$ is the projection of an element of the surface on a plane perpendicular to OP.

The integral involved in these expressions has been already evaluated and gives when the point Q is not near to the curve of contact of the tangent cone from O to the surface of the obstacle

$$X = -\iota\kappa^2\nu_1\lambda_1\,e^{\iota\kappa(Vt-R_1-R)}/(R_1+R),$$

$$Y = -\iota\kappa^2\nu_1\mu_1\,e^{\iota\kappa(Vt-R_1-R)}/(R_1+R),$$

$$Z = \iota\kappa^2(\lambda_1^2+\mu_1^2)\,e^{\iota\kappa(Vt-R_1-R)}/(R_1+R);$$

that is

$$X = -X', \qquad Y = -Y', \qquad Z = -Z',$$

and therefore at points inside the boundary of the geometrical shadow and not close to it the principal part of the electric force vanishes and consequently the principal part of the magnetic force vanishes. In the second case when OQ, QP made equal angles with the normal at Q the coefficients

$$-\nu_2(\lambda_1+\lambda_2)(l\lambda_1+m\mu_1+n\nu_1)+n\lambda_2(\lambda_1\lambda_2+\mu_1\mu_2+\nu_1\nu_2)-n\lambda_1+(l\mu_1-m\lambda_1)(\mu_2\nu_1-\mu_1\nu_2),$$

$$-\nu_2(\mu_1+\mu_2)(l\lambda_1+m\mu_1+n\nu_1)+n\mu_2(\lambda_1\lambda_2+\mu_1\mu_2+\nu_1\nu_2)-n\mu_1+(l\mu_1-m\lambda_1)(\nu_2\lambda_1-\nu_1\lambda_2),$$

$$\{\lambda_2(\lambda_1+\lambda_2)+\mu_2(\mu_1+\mu_2)\}(l\lambda_1+m\mu_1+n\nu_1)+n\nu_2(\lambda_1\lambda_2+\mu_1\mu_2+\nu_1\nu_2)-n\nu_1 \\ +(l\mu_1-m\lambda_1)(\lambda_2\mu_1-\lambda_1\mu_2),$$

vanish at Q and therefore the principal parts of X, Y, Z vanish at P, provided the point Q is not near to the curve of contact of the tangent cone. These results might have been obtained by observing that when the limits of the integral can be taken infinite the effect is the same as if the surface on which this distribution of electric and magnetic currents is placed surrounded the source and therefore at a point inside this surface the effect of the distribution is zero and at a point outside it equal and opposite to that of the source.

When the point P is inside the boundary of the geometrical shadow and so near to it that the limits of integration cannot be taken to be infinite the values of X, Y, Z as in the case of the conducting surface are given by

$$\mathrm{X} = -(1-\mathrm{L})\,\mathrm{X}', \qquad \mathrm{Y} = -(1-\mathrm{L})\,\mathrm{Y}', \qquad \mathrm{Z} = -(1-\mathrm{L})\,\mathrm{Z}',$$

where

$$\mathrm{L} = 2^{-1/2} e^{1/4\pi\iota} \int_{u_0}^{\infty} e^{-1/2\pi\iota u^2}\, du,$$

and

$$u_0 = \left\{\frac{2}{\lambda}\left(\frac{1}{\mathrm{OM}} + \frac{1}{\mathrm{PM}}\right)\right\}^{1/2} \mathrm{RM},$$

where the letters have the same signification as in the case of the conducting surface. Hence, at points inside the boundary of the geometrical shadow, the components of the electric force are

$$\mathrm{LX}', \quad \mathrm{LY}', \quad \mathrm{LZ}',$$

where X′, Y′, Z′ are the components of the electric force at the point due to the oscillator when there is no obstacle, and the components of the magnetic force are

$$\mathrm{L}\alpha', \quad \mathrm{L}\beta', \quad \mathrm{L}\gamma',$$

where $\alpha'$, $\beta'$, $\gamma'$ are the components of the magnetic force due to the oscillator. In this case these values of the components of the magnetic and electric forces are valid up to the boundary of the obstacle, therefore the principal parts of the components of the magnetic and electric forces at a point P inside the boundary of the geometrical shadow are the same whether the obstacle is perfectly conducting or perfectly absorbing, provided the point P is at a distance from the surface of the obstacle measured along OP of a higher order than $\rho\,(\kappa\rho)^{-1/3}$, where $\rho$ is the radius of curvature of the section of the surface through the tangent in the plane of incidence, and the value of the principal part of the magnetic force tangential to the surface of the obstacle at a point on it inside the boundary of the geometrical shadow when the obstacle is perfectly conducting is double the value it has when the obstacle is perfectly absorbing; the value of the principal part of the component of the electric force normal to the surface of the obstacle at a point inside the boundary of the geometrical shadow when the obstacle is perfectly conducting is also double the value it has when the obstacle is perfectly absorbing.

When the point P is near to the boundary of the geometrical shadow and outside it the principal parts of the components of the electric and magnetic forces at the point P are

$$\mathrm{LX}', \quad \mathrm{LY}', \quad \mathrm{LZ}', \quad \mathrm{L}\alpha', \quad \mathrm{L}\beta', \quad \mathrm{L}\gamma',$$

where

$$\mathrm{L} = 2^{-1/2} e^{1/4\pi\iota} \int_{-u_0}^{\infty} e^{-1/2\pi\iota u^2}\, du,$$

and $u_0$ has the same meaning as above.

## *An Imperfectly Conducting Obstacle.*

When the obstacle is an imperfectly conducting body the conditions to be satisfied at the surface of the obstacle are that the components of the electric and magnetic forces tangential to the surface are continuous. The electric and magnetic current distributions to be placed on the surface to effect this can be obtained from the following considerations. At a point on the surface of the obstacle not near to the point where the axis of the oscillator meets it the incident waves may be treated as if they were plane waves; let $\mathrm{M}_1$ and $\mathrm{M}_2$ be the components of the magnetic force in the incident waves tangential to the surface, $\mathrm{M}_1$ being in the plane of incidence and $\mathrm{M}_2$ perpendicular to it, and let $\mathrm{E}_1$ and $\mathrm{E}_2$ be the components of the electric force in the incident waves tangential to the surface, $\mathrm{E}_1$ being in the plane of incidence and $\mathrm{E}_2$ perpendicular to it, then, if $\phi$ is the acute angle between the direction of the incident waves and the normal to the surface at a point Q on it, writing

$$\epsilon = (\kappa' + \iota\kappa \cos\phi)/(\kappa' - \iota\kappa\cos\phi), \qquad \epsilon' = (\kappa'\cos\phi + \iota\kappa)/(\kappa'\cos\phi - \iota\kappa),^{*}$$

where

$$\kappa'^2 = \kappa^2 \sin^2\phi + 4\pi\iota\kappa \mathrm{V}/\sigma,$$

and $\sigma$ is the specific resistance of the obstacle, the electric current distribution to be placed on the surface has components

$$\mathrm{M}_2(1+\epsilon'), \quad \mathrm{M}_1(1+\epsilon),$$

and the magnetic current distribution to be placed on the surface has components

$$\mathrm{E}_2(1-\epsilon), \quad \mathrm{E}_1(1-\epsilon').$$

The effect of this distribution can be immediately obtained from the previous calculations by superposing the two distributions

$$2\mathrm{M}_2\epsilon', \quad 2\mathrm{M}_1\epsilon,$$

and the two distributions

$$\mathrm{M}_2(1-\epsilon'), \quad \mathrm{E}_1(1-\epsilon'); \qquad \mathrm{M}_1(1-\epsilon), \quad \mathrm{E}_2(1-\epsilon);$$

the first two involve the same calculation as in the case of the perfectly conducting

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[*] The general value of $\epsilon'$ is $(4\pi\mathrm{V}/\sigma + \kappa'\sec\phi)/(4\pi\mathrm{V}/\sigma - \kappa'\sec\phi)$, which has the approximate value in the text when the wave-length $\lambda$ of the oscillations is great compared with $\sigma/\mathrm{V}$.

obstacle, and the second two involve the same calculation as in the case of the perfectly absorbing obstacle, hence, at points inside the tangent cone from the oscillator to the surface of the obstacle, the principal parts of the components of the electric and magnetic forces vanish when the point is not near to the boundary of the geometrical shadow, at a point P outside the boundary of the geometrical shadow and not near to it, the principal parts of the components of the electric and magnetic forces in the reflected waves at P are given by multiplying the components due to $2M_2$ at Q in the case of perfect reflection by $\epsilon'$ and the components due to $2M_1$ at Q by $\epsilon$, where OQ, QP make equal angles with the normal at Q to the surface. When the point P is inside the tangent cone and near to the boundary of the geometrical shadow, the distribution $2M_2\epsilon'$ gives the corresponding components multiplied of $-(1-L)\epsilon'$ and the distribution $M_2(1-\epsilon')$, $E_1(1-\epsilon')$ the corresponding components multiplied by $-(1-L)(1-\epsilon')$, that is these two distributions give the corresponding components multiplied by $-(1-L)$ and the other two distributions give the corresponding components multiplied by $-(1-L)$; therefore the principal parts of the components of the electric and magnetic forces at the point P due to the oscillator and to the assumed distribution on the surface are

$$LX', \quad LY', \quad LZ', \quad L\alpha', \quad L\beta', \quad L\gamma',$$

where $X'$, $Y'$, $Z'$, $\alpha'$, $\beta'$, $\gamma'$ are the principal parts of the components of the electric and magnetic forces due to the oscillator at the point P and L has the value defined above. It may be verified as in the case of the perfect conductor that the boundary conditions at points on the surface of the obstacle not near to the curve of contact of the tangent cone from the oscillator are satisfied by this assumed distribution on the surface. It also follows from the investigation for the case of the perfectly conducting obstacle that the principal parts of the components of the electric and magnetic forces at a point P inside the boundary of the geometrical shadow are given by

$$LX', \quad LY', \quad LZ', \quad L\alpha', \quad L\beta', \quad L\gamma',$$

provided P is at a distance from the surface of the obstacle which is of an order higher than $\rho(\kappa\rho)^{-1/3}$ and the perpendicular distance of OP from the point R on the curve of contact of the tangent cone from O to the surface is of the same or higher order than $\rho(\kappa\rho)^{-2/3}$, where $\rho$ is the radius of curvature of the normal section of the surface through OR. The principal parts of the components of the electric and magnetic forces at points on the surface of the obstacle near the curve of contact but at a distance along the surface from it of higher order than $\rho(\kappa\rho)^{-1/3}$ can be determined as follows. Let $M'_1$, $M'_2$ be the principal parts of the components of the magnetic force due to the oscillator tangential to the surface at the point P on it inside the boundary of the geometrical shadow, $M'_1$ being in the plane containing OP and the normal to the surface at P, and $M'_2$ perpendicular to this plane. Further, let $M_1$, $M_2$

be the principal parts of the components of the actual magnetic force at P in these directions, then P being at a distance along the surface from the curve of contact of higher order than $\rho(\kappa\rho)^{-1/3}$, the principal part of the component of the magnetic force in the plane of OP and the normal to the surface at P which is due to the oscillator and the actual distribution on the part of the surface inside the tangent cone from O and nearest to O is $\mathrm{LM}'_1$, and the principal part of the corresponding component of the magnetic force due to the local distribution at P is $\epsilon\mathrm{M}_1/(1+\epsilon)$, where $\epsilon = (\kappa'+\iota\kappa\cos\phi)/(\kappa'-\iota\kappa\cos\phi)$, and $\phi$ is the acute angle between the normal at P and OP.

Therefore

$$\mathrm{M}_1 = \epsilon\mathrm{M}_1/(1+\epsilon)+\mathrm{LM}'_1,$$

that is

$$\mathrm{M}_1 = (1+\epsilon^1)\,\mathrm{LM}'_1.$$

Similarly

$$\mathrm{M}_2 = (1+\epsilon')\,\mathrm{LM}'_2,$$

where

$$\epsilon' = (\kappa'\cos\phi+\iota\kappa)/(\kappa'\cos\phi-\iota\kappa),$$

and the principal parts of the components of the electric force at P can be similarly obtained. When $\kappa$ is small compared with $4\pi\mathrm{V}/\sigma$, $\kappa'$ is approximately given by $\kappa' = \kappa_1(1+\iota)$, where $\kappa_1 = (2\pi\kappa\mathrm{V}/\sigma)^{1/2}$, the modulus of $1+\epsilon$, is greater than 2, and the modulus of $1+\epsilon'$ is greater than 2 provided $\cos\phi$ is greater than $\kappa/2\kappa_1$; therefore at a point P on the part of the surface next the oscillator the amplitude of the tangential magnetic force is greater than in the case of a perfectly conducting obstacle, and at a point on the part of the surface of the obstacle remote from the oscillator the tangential magnetic force is greater than in the case of a perfectly conducting obstacle, provided $\cos\phi$ is greater than $\kappa/2\kappa_1$. The same results hold for the components of the electric force normal to the surface of the obstacle.

It appears from the foregoing investigations that at a point P inside the boundary of the geometrical shadow formed by any opaque obstacle the principal parts of the components of the electric and magnetic forces are

$$\mathrm{LX}',\quad \mathrm{LY}',\quad \mathrm{LZ}',\quad \mathrm{L}\alpha',\quad \mathrm{L}\beta',\quad \mathrm{L}\gamma',$$

where $\mathrm{X}'$, $\mathrm{Y}'$, $\mathrm{Z}'$, $\alpha'$, $\beta'$, $\gamma'$ are the principal parts of the components of the electric and magnetic forces at the point P due to the oscillator, when the point P is at a distance measured along OP from the surface of the obstacle of a higher order than $\rho(\kappa\rho)^{-1/3}$, that when the obstacle is perfectly absorbing the principal parts of the components of the electric and magnetic forces at a point P on the surface of the obstacle inside the boundary of the geometrical shadow have the above values, that when the obstacle is perfectly conducting the ratio of the electric force normal to the surface at the point P on it to the electric force normal to the surface of a perfectly absorbing obstacle occupying the same space at the same point on it is 2, and that when the obstacle is imperfectly conducting this ratio is greater than 2,* but decreases as the distance along the surface of the point P from the edge of the shadow increases when the magnetic force in the incident waves is perpendicular to the plane of incidence.

The same methods as have been applied in the foregoing can be used to solve the problem of the case of a transparent obstacle; it is convenient in this case to treat separately the components of the electric and magnetic forces tangential to the surface in and perpendicular to the plane of incidence.

[*October 17.*—The investigation given above of the effect of a perfectly absorbing obstacle assumes that the electric and magnetic forces on the surface change abruptly at the curve of contact of the tangent cone from the point O to the surface. When the creeping effect at the edge is taken into account, the quantity L gives the ratio of the forces at a point P inside the geometrical shadow to the forces at that point due to the oscillator alone, only if the point P is subject to the same restrictions as in the case of the perfectly conducting obstacle, that the distance of the point P along OP from the surface of the obstacle is of higher order than $\rho(\kappa\rho)^{-1/3}$, and that OM is of higher order than $\rho(\kappa\rho)^{-1/3}$. The theorem, that the electric force normal to the surface of the obstacle at a point on it when the obstacle is perfectly conducting has double the value it has when the obstacle is perfectly absorbing, can be established generally as follows: Let E and M be the electric and magnetic forces in the incident waves tangential to the surface of the obstacle at a point on it. If the surface is perfectly conducting there is an electric current distribution $2\mathrm{M}\omega$ on it and a zero magnetic current distribution, and if the obstacle is incapable of supporting magnetic force there is a zero electric current distribution on it and a magnetic current distribution $2\mathrm{E}\omega'$; the superposition of these two distributions gives the solution for the case when the obstacle is perfectly absorbing and the electric and magnetic forces in the incident waves tangential to the surface at a point on it are 2E, 2M. Hence the electric current distribution on the surface of the obstacle when it is perfectly absorbing is $\mathrm{M}\omega$, and when it is perfectly conducting $2\mathrm{M}\omega$, the electric and magnetic forces in the incident waves tangential to the surface being E, M; and therefore the magnetic force tangential to the surface and the electric force normal to the surface of the obstacle when it is perfectly conducting have each double the value they have when the obstacle is perfectly absorbing. Similar reasoning applies to the case of the imperfectly conducting obstacle, the ratio in this case being $1+\epsilon$ for the component of magnetic force tangential to the surface in the plane of incidence and $1+\epsilon'$ for the component perpendicular to the plane of incidence.]

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* For the case of waves of the wave-lengths used in wireless telegraphy, the conducting body being the sea, the value of this ratio at a distance of 150 miles is not greater than 2·06 and decreases as the distance increases.